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M334 PART IV

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

PART IV CALCULUS ON A SURFACE

DIFFERENTIAL GEOMETRY

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

DIFFERENTIAL GEOMETRY PART IV

CALCULUS ON A SURFACE

A commentary on Chapter IV of O'Neill's
Elementary Differential Geometry
Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

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Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition (W.A. Benjamin/Addison-Wesley, 1973).

R.C. Smith and P. Smith: *Mechanics,* SI Edition (John Wiley, 1972).

Conventions

Before starting work on this text, please read M334 Part Zero. Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory* by E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P.Smith.

Reference to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

INTRODUCTION

This chapter uses the differential and vector calculus developed in Sections I.1 to II.1.

It introduces the second major theme of this course, *surfaces.* The surfaces in which we shall be interested are subsets of E^3 that locally look like some open region of E^2. You have already met some of these, such as planes, spheres and cylinders.

After discussing several ways of describing a surface we extend the definitions of *tangent vector, differential form* and *mapping* to apply to them.

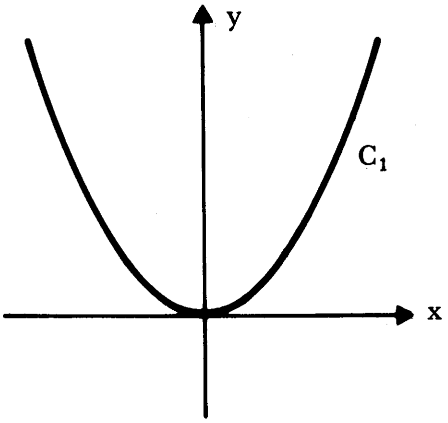
Finally we use differential forms to describe orientability of a surface.

IV.1 SURFACES IN E^3

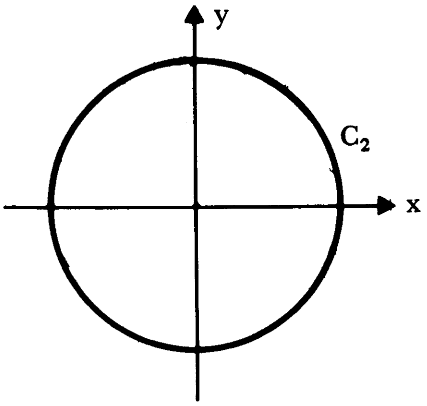
Introduction

This section follows on from Sections I.2, I.4, I.5, I.7 and II.1.

We shall be using implicitly defined curves in the plane. You first met these in Section I.4. Typical examples are:



(a) $C_1 : y - x^2 = 0$



(b) $C_2 : x^2 + y^2 = 1$

In general a curve is defined as a subset C: g = c, where g is a differentiable function and c is a constant. We shall always assume that, for each point (x_0, y_0) in C, either $(\partial g/\partial x)\ (x_0, y_0) \neq 0$ or $(\partial g/\partial y)\ (x_0, y_0) \neq 0$, and that if C has more than one component then we are restricting attention to just one of them. With these restrictions the only curves that we obtain are similar to C_1 and C_2.

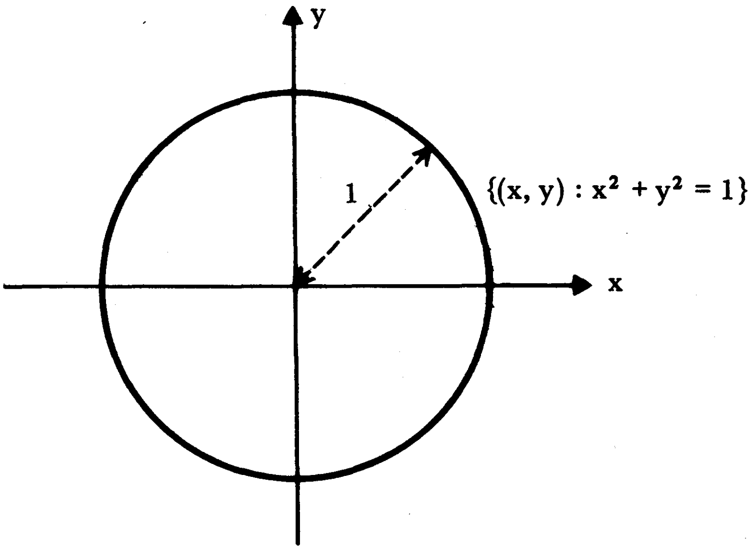
(a) Given a curve like C_1 we can parametrize it by a regular differentiable function α from R to E^2 that is one-to-one and whose inverse map α^{-1} from C_1 to R is continuous. In our example of C_1 such a parametrization is given by $\alpha(t) = (t, t^2)$. This has inverse $\alpha^{-1}(x, y) = x$ for points in C_1.

(b) Given a curve like C_2 we can parametrize it by a periodic regular differentiable function β from R to E^2. We can also demand that if we restrict attention to small enough open subsets of C_2 they will be the one-to-one images of small intervals of R and the inverse functions between these small subsets will be continuous. In our example of C_2 such a parametrization is given by $\beta(t) = (\cos t, \sin t)$. A typical 'local' inverse for this function is given by $\beta^{-1}(x, y) = \cos^{-1} x$ for all those points p in C_2 with $y(p) > 0$.

We shall need the implicit function theorem, a two dimensional form of which was dealt with in an appendix to *Unit M231, Applications of the Derivative,* Section 8.

As an example of an implicitly defined curve in E^2 we can consider again the unit circle

$$C: f(x, y) = x^2 + y^2 = 1.$$



We can use the implicit function theorem to describe the local behaviour of this curve. The implicit function theorem states:

Theorem IV.1.A

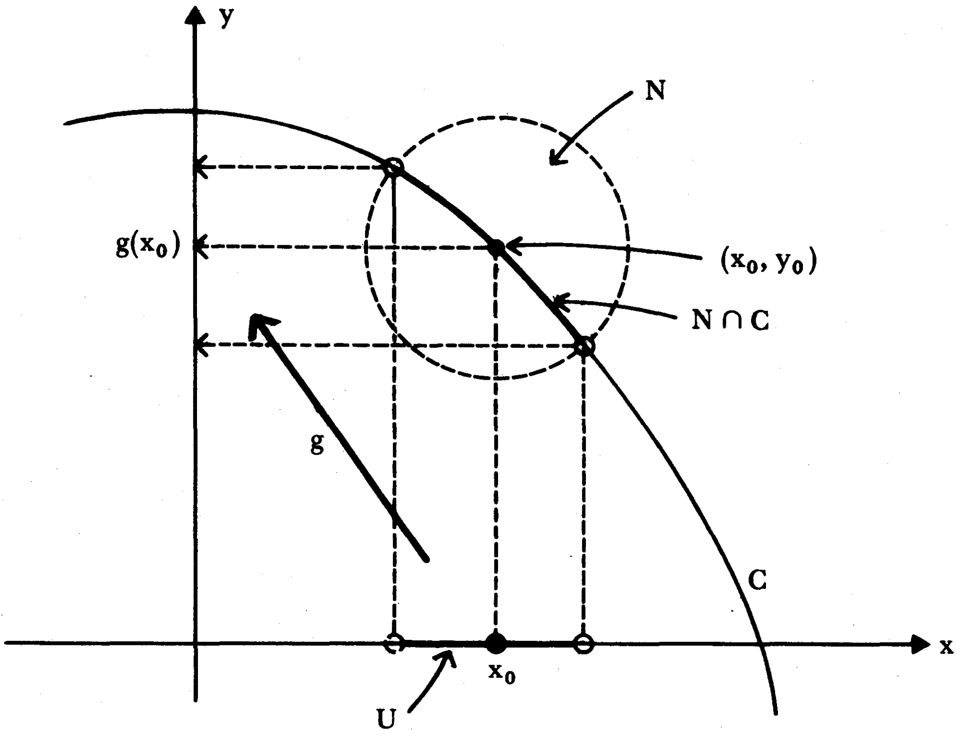
If the point (x_0, y_0) belongs to C and if $(\partial f/\partial y)(x_0, y_0) \neq 0$ then there exists:

(i) a neighbourhood U of x_0 in the real line;

(ii) a unique differentiable function $g : U \longrightarrow R$; and

(iii) a neighbourhood N of (x_0, y_0) in the plane; such that

$$N \cap C = \{(t, g(t)) : t \in U\}.$$



That is, locally the curve C can be described as the graph of a mapping $g : U \longrightarrow R$. Similarly we can describe C equally well in terms of y near (x_0, y_0) if $(\partial f/\partial x)(x_0, y_0) \neq 0$.

In our example, if $x_0 = \frac{1}{2}$ we can choose for U the open interval $(\frac{1}{4}, \frac{3}{4})$ and then g is the function

$$g : t \longmapsto (1 - t^2)^{\frac{1}{2}}.$$

For a suitably chosen N we have that

$$N \cap C = \{(t, (1 - t^2)^{\frac{1}{2}}) : \tfrac{1}{4} < t < \tfrac{3}{4}\};$$

that is, N ∩ C is the graph of g.

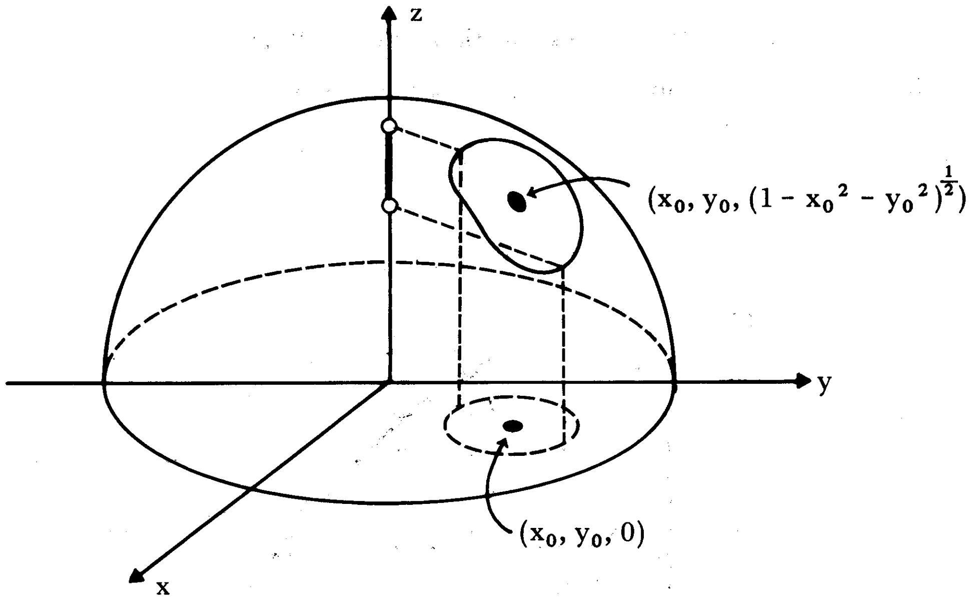
In this section we shall apply an extension of the implicit function theorem to implicitly defined subspaces of E^3. For instance, near a point (x_0, y_0, z_0) on the unit sphere

$$\Sigma : f(x, y, z) = x^2 + y^2 + z^2 = 1$$

for which

$$(\partial f/\partial z)(x_0, y_0, z_0) = 2z_0$$

is nonzero, the sphere can be described as the graph of a function g, where $g(x, y) = z = (1 - x^2 - y^2)^{\frac{1}{2}}$.

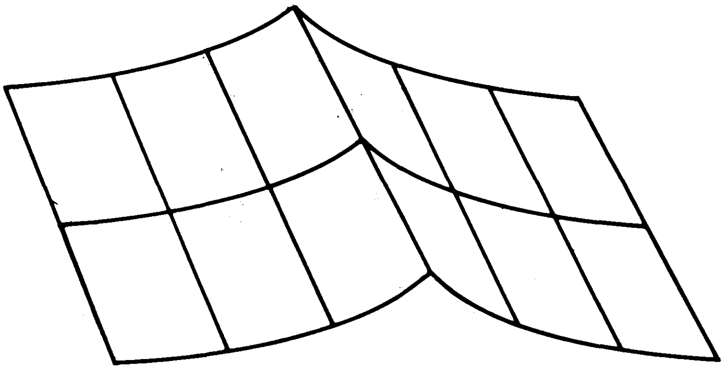


READ: Section IV.1 (pages 124 - 131).

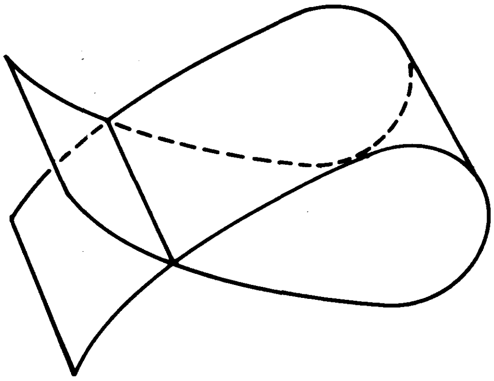
Comments

(i) *The plane E^2* When we use E^2, or some open set D in E^2, as the domain for a patch **x** we write u and v for the natural coordinate functions rather than x and y or x_1 and x_2. Consequently when we want to refer to a variable point in E^2 we write (u, v), though strictly speaking this is the identity function from E^2 to itself. Whichever interpretation you adopt it is unlikely that any ambiguity will arise. When we have to use the partial derivatives of a function f whose domain is in E^2 we write them as $\partial f/\partial u$ and $\partial f/\partial v$.

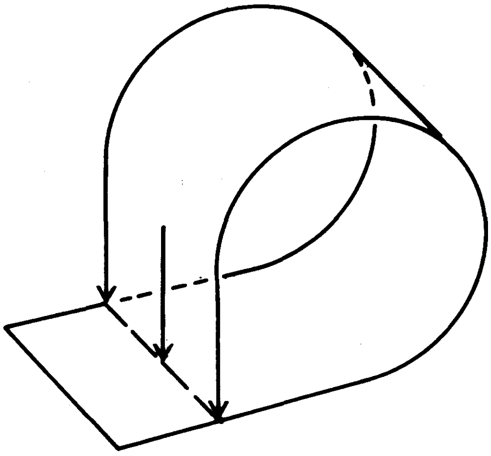
(ii) *Page 124: Proper patches* The conditions in the definition of a proper patch are there to rule out certain pathological possibilities. The condition that the mapping **x** is regular means that its derivative $\mathbf{x}_*$ is one-to-one and hence no nonzero tangent vector is mapped to a zero tangent vector. Since derivatives preserve velocities, each curve with nonzero velocity will be mapped to another curve with nonzero velocity. As we saw in Section I.4 this means that **x** does not introduce any kinks into the patch. The image of a patch cannot look like the following diagram.



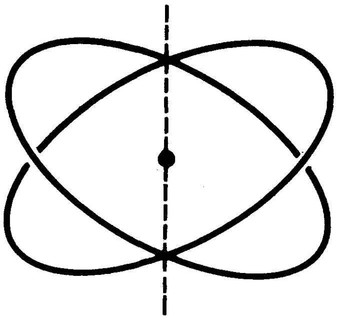
The condition that **x** is one-to-one rules out the possibility that the patch might intersect itself, as in the following diagram.



The condition that $\mathbf{x}^{-1}$ is continuous rules out the possibility that the patch might approach itself arbitrarily closely, as in the following diagram.

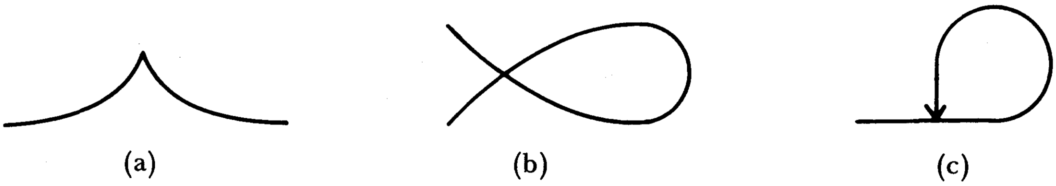


(iii) *Page 125: Definition 1.2* To prove that a subset of E^3 is not a surface we need to find only one point that does not have any neighbourhoods that are like smooth open discs. One way to do this is to look at the boundaries of suitably small neighbourhoods to see if they are like circles. For instance in the second example of Comment (ii) the boundary of a typical small neighbourhood of a point on the line of self-intersection is shown in the following diagram.



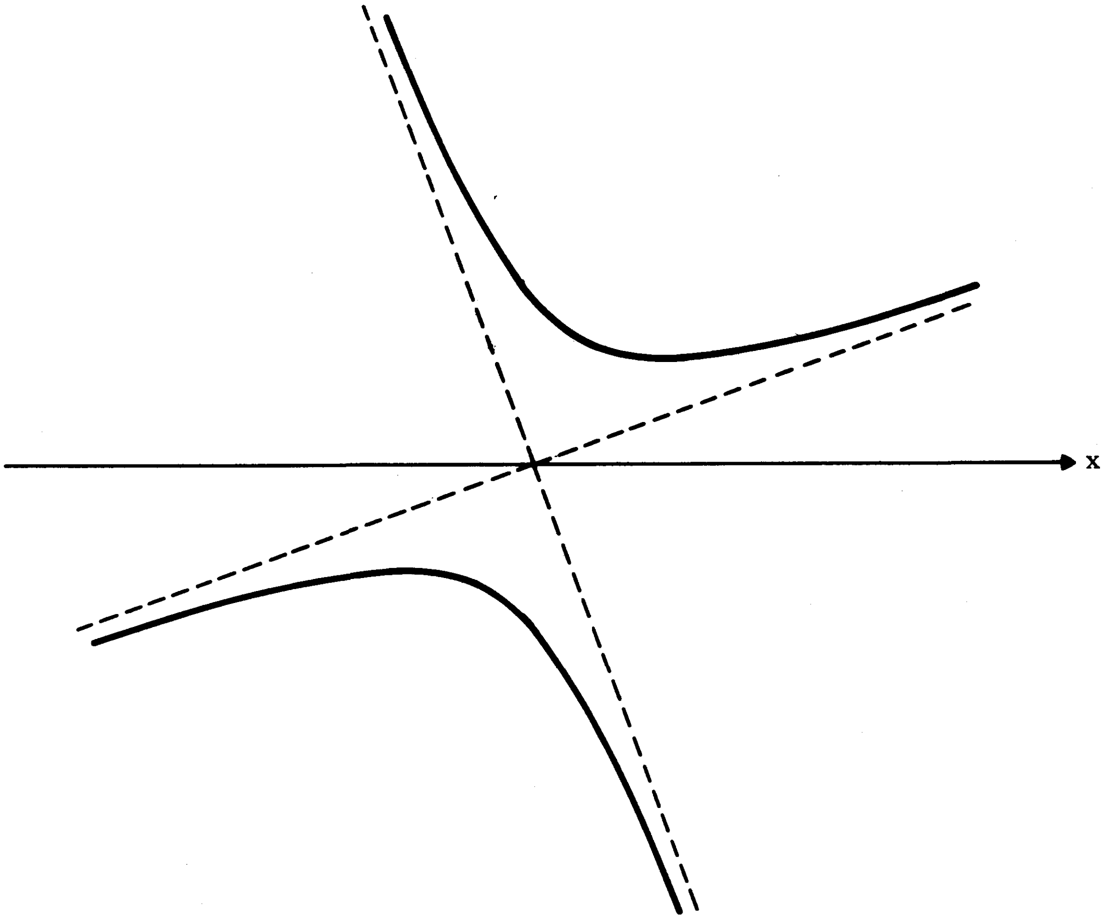
We shall assume that our intuition does enable us to distinguish this from a circle.

(iv) *Page 129: Curves and surfaces* The curves used in both Example 1.5 and Example 1.6 are defined implicitly. As in Section I.4 this rules out the possibility that they may be of any of the following forms.



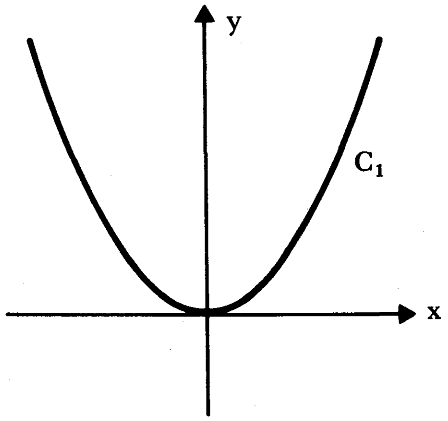
This in turn ensures that the surfaces derived from them, as in Examples 1.5 and 1.6, are not pathological in any of the ways mentioned in Comment (ii). In fact the implicit function theorem discussed in the introduction proves that the above types of curve cannot occur. Curves (b) and (c) cannot be expressed locally as graphs and curve (a) cannot be expressed as the graph of a differentiable function.

When dealing with implicitly defined curves we shall always restrict ourselves to one component at a time. In Example 1.6 this enables us to assume that the curve is in the upper half plane. If we did not have this restriction we would have to consider curves like the following hyperbola.

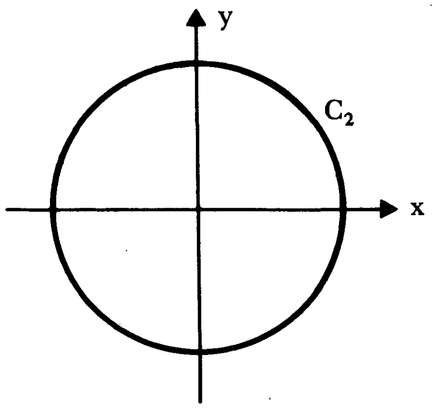


When this is revolved about the x axis the spaces traced out by the two components intersect and the result is not a surface.

As mentioned in the introduction this restriction limits us to two types of curve. The following are typical examples. We can revolve them about any line that does not intersect them.



(a) $C_1 : y - x^2 = 0$

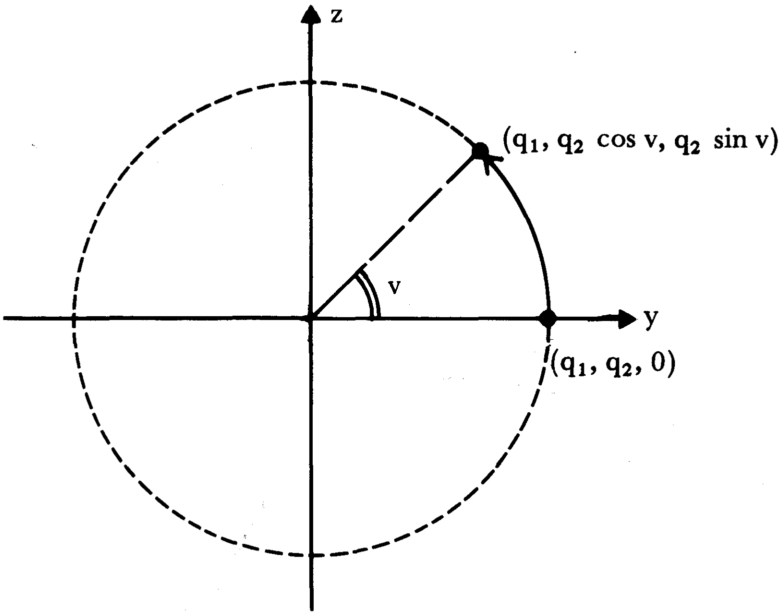


(b) $C_2 : x^2 + y^2 = 1$

For curves like C_1 we shall use only one-to-one parametrizations and with curves like C_2 we shall use only periodic parametrizations. As usual all parametrizations of implicitly defined curves will be regular.

Supplementary Comments

(i) *Page 129: Example 1.6* Rotating about the x axis we keep the x coordinate fixed.



Hence q_2 gives the radius of the circle that the point $(q_1, q_2, 0)$ traces out as it is revolved. When it has been revolved through an angle v its new coordinates are $(q_1, q_2 \cos v, q_2 \sin v)$.

If the point $p = (p_1, p_2, p_3)$ is revolved about the x axis, its x coordinate remains unchanged and its distance from the x axis remain unchanged. This distance is $(p_2^2 + p_3^2)^{\frac{1}{2}}$. Hence when it has been swung round into the upper half of the xy plane it reaches the point $\bar{p} = (p_1, (p_2^2 + p_3^2)^{\frac{1}{2}}, 0)$. The point p is on the surface of revolution if and only if this point is on the original curve. So (p_1, p_2, p_3) belongs to M if and only if $f(p_1, (p_2^2 + p_3^2)^{\frac{1}{2}}) = c$. That is, M is given by $M : g = c$ where $g(x, y, z) = f(x, (y^2 + z^2)^{\frac{1}{2}})$.

How do we use the chain rule to prove that dg is never zero? Firstly, since g is a mapping from E^3 to R

$$dg(v) = v[g] = \frac{d}{dt}(g(p + tv))\bigg|_{t=0} = g_*(v)$$

for any tangent vector v to E^3 at p. Similarly for any tangent vector w to E^2 at q then

$$df(w) = f_*(w),$$

where f is a function from E^2 to R and the differential df is defined by analogy with the definition for mappings from E^3 to R. Now g is the composite mapping $f \circ F$ where

$$F(p_1, p_2, p_3) = (F_1(p_1, p_2, p_3), F_2(p_1, p_2, p_3))$$
$$= (p_1, (p_2^2 + p_3^2)^{\frac{1}{2}}).$$

Hence by the chain rule

$$dg(v) = g_*(v) = f_*(F_*(v)) = df(F_*(v)).$$

Now the Jacobian matrix for F is

$$F_* = \begin{pmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{y}{(y^2+z^2)^{\frac{1}{2}}} & \dfrac{z}{(y^2+z^2)^{\frac{1}{2}}} \end{pmatrix}$$

Hence at points of M the mapping F_* is onto, has rank 2, since either y or z is nonzero as M does not intersect the x axis.

Assume dg is zero at p on M; then dg(v) = 0 for all vectors v tangent to M at p. Now if w is tangent vector to E^2 at F(p): then, since F_* is onto, $w = F_*(v)$ for some tangent vector v to E^3 at p. Hence

$$df(w) = df(F_*(v)) = dg(v) = 0,$$

by assumption, and so df is zero at F(p). Now for mappings from E^2 to R the analogue of Corollary I.5.5 gives

$$df = \frac{\partial f}{\partial u}\,du + \frac{\partial f}{\partial v}\,dv.$$

Hence $\partial f/\partial u$ and $\partial f/\partial v$ are both zero at F(p). But

$$F(p) = (p_1, (p_2{}^2 + p_3{}^2)^{\frac{1}{2}})$$

is a point of the profile curve C and so by the definition of an implicitly defined curve $\partial f/\partial u$ and $\partial f/\partial v$ cannot both be zero. This contradiction tells us that our original assumption that dg was zero at p is false; dg is never zero on M.

Summary

Notation

x	Page 124, Definition 1.1
M	Page 125, Definition 1.2
Σ	Page 125, line −6
M : z = f(x, y)	Page 127, Example 1.3
M : g = c	Page 127, line −11

Definitions

(i)	Coordinate patch **x** : D⟶E^3	Page 124, Definition 1.1
(ii)	Proper patch **x** : D⟶E^3	Page 124, line −2
(iii)	Surface M	Page 125, Definition 1.2
(iv)	Monge patch **x**(u, v) = (u, v, f(u, v))	Page 127, line 8
(v)	Simple surface M = **x**(D)	Page 127, line 11
(vi)	Graph of a function M : z = f(x, y)	Page 127, Example 1.3
(vii)	Implicitly defined surface M : g = c	Page 127, line −11
(viii)	Cylinder	Page 129, Example 1.5
(ix)	Surface of revolution	Page 129, Example 1.6
(x)	Parallel	Page 130, line 8
(xi)	Meridian	Page 130, line 9

Examples

(i)	Unit sphere Σ	Page 125, line -3
(ii)	Hemispherical patches on the sphere, $x(u, v) = (u, v, \pm(1-u^2-v^2)^{\frac{1}{2}})$	Page 126, line 7
(iii)	Sphere	Page 128, line -2
(iv)	Circular cylinder, $M : x^2 + y^2 = r^2$ in E^3	Page 129, line -13

Results

(i)	Graphs are surfaces.	Page 127, Example 1.3
(ii)	An implicitly defined subset $M : g = c$ is a surface if the differential dg is never zero at any point of M.	Page 127, Theorem 1.4
(iii)	Cylinders are surfaces.	Page 129, Example 1.5
(iv)	Surfaces of revolution are surfaces.	Page 129, Example 1.6

Techniques

(i)	Recognizing a (proper) patch.	Page 124, Definition 1.1 (and Page 124, line -2)
(ii)	Determining whether subsets of E^3 are surfaces by looking for proper patches.	Page 125, Definition 1.2
(iii)	Recognizing that a subset of E^3 is a surface since it is a graph, a cylinder or a surface of revolution.	Page 127, Example 1.3, Page 129, Example 1.5, Page 129, Example 1.6
(iv)	Determining whether an implicitly defined subset of E^3, $M : g = c$, is a surface by evaluating the differential dg.	Page 127, Theorem 1.4
(v)	Sketching surfaces.	

Exercises

Technique (i)

1. Page 132, Exercise 4.

Techniques (ii) and (iii)

2. Page 131, Exercise 1. Deleting as few points as possible from each subset construct a surface. Explain why it is a surface.

Technique (iv)

3. Page 132, Exercise 5.

Technique (v)

4. Page 132, Exercise 6.

Solutions

1. Page 132, Exercise 4.

The Jacobian matrix for the mapping **x** = $(x_1, x_2, x_3) : E^2 \longrightarrow E^3$ is

$$\begin{pmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_1}{\partial v} \\ \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_2}{\partial v} \\ \dfrac{\partial x_3}{\partial u} & \dfrac{\partial x_3}{\partial v} \end{pmatrix}$$

(a) The Jacobian matrix is

$$\begin{pmatrix} 1 & 0 \\ v & u \\ 0 & 1 \end{pmatrix}$$

This always has rank 2 and so **x** is regular.

Now if

$$\mathbf{x}(u, v) = \mathbf{x}(u', v')$$

then

$$(u, uv, v) = (u', u'v', v')$$

and so

$$u = u', v = v'.$$

Hence **x** is one-to-one.

Since **x** is one-to-one and regular it is a patch.

(b) The Jacobian matrix is

$$\begin{pmatrix} 2u & 0 \\ 3u^2 & 0 \\ 0 & 1 \end{pmatrix}$$

When $u = 0$ this matrix has rank 1. Hence **x** is not regular and so is not a patch.

(c) The Jacobian matrix is

$$\begin{pmatrix} 1 & 0 \\ 2u & 0 \\ 0 & 1 + 3v^2 \end{pmatrix}$$

and since $v^2 \geqslant 0$ it has rank 2. Hence **x** is regular.

Suppose $\mathbf{x}(u, v) = \mathbf{x}(u', v')$.

Then

$$(u, u^2, v + v^3) = (u', u'^2, v' + v'^3).$$

Hence

$$u = u' \quad \text{and} \quad (v - v') + (v^3 - v'^3) = 0.$$

It follows that

$$(v - v')\,(1 + (v^2 + vv' + v'^2)) = 0$$

and hence

$$(v - v')\,(1 + (v + \frac{v'}{2})^2 + \frac{3}{4}v'^2) = 0.$$

Since $1 + (v + \frac{v'}{2})^2 + \frac{3}{4}v'^2 > 0$ it follows that $v = v'$ also and hence **x** is one-to-one. Since **x** is one-to-one and regular it is a patch.

(d) If $\mathbf{x}(u, v) = \mathbf{x}(u', v')$ then

$$\cos 2\pi u = \cos 2\pi u', \quad \sin 2\pi u = \sin 2\pi u' \quad \text{and} \quad v = v'.$$

These conditions are satisfied whenever $u = u' + n$, for any integer n, and $v = v'$. Hence the mapping is not one-to-one and **x** is not a patch.

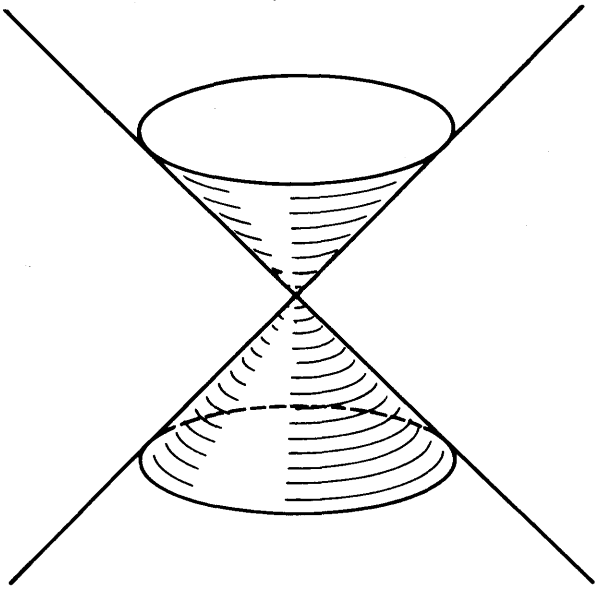
2. Page 131, Exercise 1.

(a) We can write M as $M : g = 0$, where $g(x, y, z) = z^2 - x^2 - y^2$.
By Corollary I.5.5 the differential of g is

$$dg = -2x\,dx - 2y\,dy + 2z\,dz$$

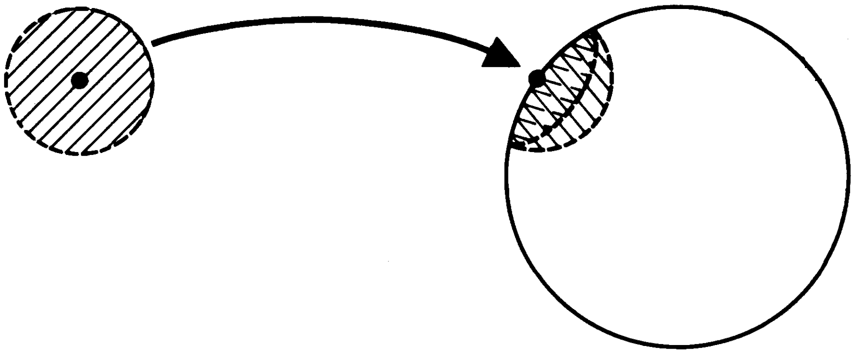
and this is zero only at the origin. Hence, by Theorem IV.1.4, the cone minus its apex is a surface.

Since the boundary of a small neighbourhood of the origin is two circles we cannot include the origin if we require a surface.

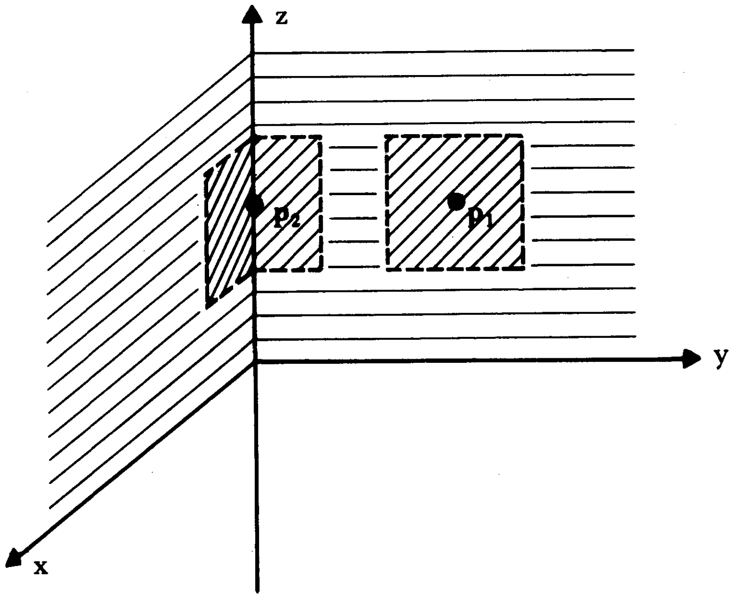


(b) Points not on the boundary of the closed disc have the same small neighbourhoods as if they were considered as points of the plane $P : z = 0$. Since the differential dz is nonzero it follows from Theorem IV.1.4 that the disc minus its boundary is a surface.

If we try to map a small open disc continuously to the closed disc in such a way that the centre of the open disc is mapped to a point on the boundary of the closed disc we find that we have to fold it over. Hence the map cannot be one-to-one and so the boundary points cannot be included in the surface.



(c)



The region $M : xy = 0,\ x \geqslant 0,\ y \geqslant 0$ can be written as the union of two half planes $M_1 : x = 0,\ y \geqslant 0$ and $M_2 : y = 0,\ x \geqslant 0$. For any point p_1 in the interior of either of these planes it is easy to find a rectangular coordinate patch. Thus the subset of M obtained by deleting the z axis is a surface. However, for any point p_2 on the z axis, the intersection of these two half planes, it is impossible to find such a patch. Any neighbourhood of p_2 has a right angle bend in it and so no mapping $x : D \longrightarrow E^3$ onto a neighbourhood of p_2 will be differentiable and regular at p_2.

3. Page 132, Exercise 5.

(a) By Theorem IV.1.4, M : $(x^2 + y^2)^2 + 3z^2 = 1$ is a surface if dg is never zero on M, where $g(x, y, z) = (x^2 + y^2)^2 + 3z^2$. Now

$$dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz, \quad \text{by Corollary 1.5.5,}$$

and hence

$$\begin{aligned} dg &= 2(x^2 + y^2)\, 2x\, dx + 2(x^2 + y^2)\, 2y\, dy + 6z\, dz \\ &= 4(x^2 + y^2)\, (x\, dx + y\, dy) + 6z\, dz. \end{aligned}$$

Hence dg is zero only at the origin and since this is not a point of M it follows that M is a surface.

(b) By Theorem IV.1.4, M : $z(z - 2) + xy = c$ is a surface if dg is never zero on M, where $g(x, y, z) = z(z - 2) + xy$.

Now

$$dg = 2(z - 1)\, dz + x\, dy + y\, dx$$

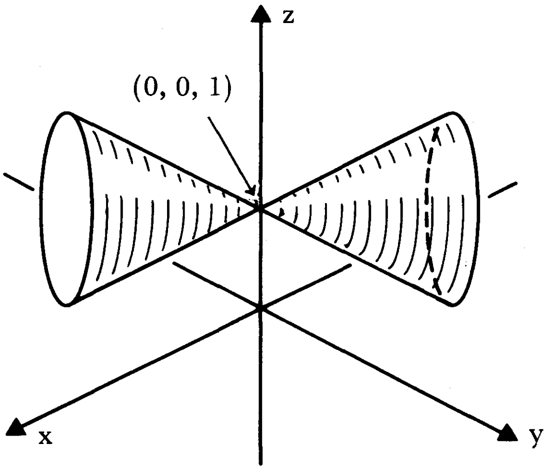
and hence dg is zero only at the point (0, 0, 1).

The point (0, 0, 1) belongs to M if and only if

$$\begin{aligned} &(z(z - 2) + xy)\,(0, 0, 1) = c \\ &1(1 - 2) + 0.0 = c \\ &\qquad c = -1. \end{aligned}$$

Hence M is a surface except possibly when $c = -1$.

When $c = -1$, the set M is given by M : $z(z - 2) + xy = -1$ or, on rearranging, M : $(z - 1)^2 + xy = 0$. A sketch of the set M near the point (0, 0, 1) shows that it is a 'double' cone. In Solution 2, above, we saw that this is not a surface.



4. Page 132, Exercise 6.

M : $z = y^3 - 3yx^2$

The intersection of M with the xy plane is given by

S : $0 = y^3 - 3yx^2$, $z = 0$.

Now if

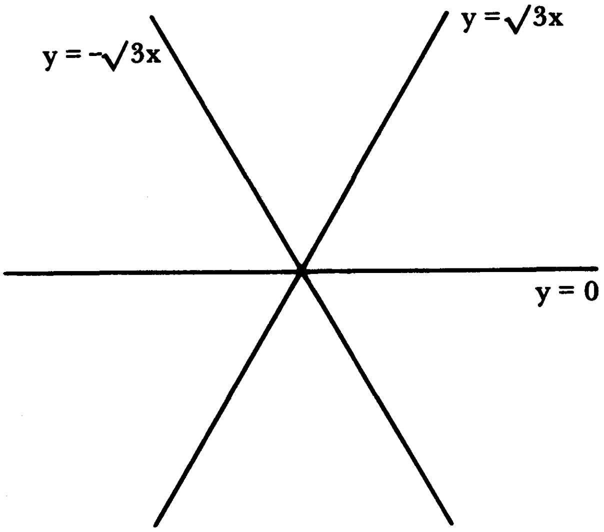
$$0 = y^3 - 3yx^2$$

then

$$\begin{aligned} 0 &= (y^2 - 3x^2)y \\ &= (y - \sqrt{3}x)(y + \sqrt{3}x)y. \end{aligned}$$

The intersection consists of three lines in the xy plane. The lines are $y = 0$, $y - \sqrt{3}x = 0$ and $y + \sqrt{3}x = 0$.

These lines make angles of $\frac{\pi}{3}$ with each other.



As a corollary to the intermediate value theorem, we can deduce that throughout each of the six resulting regions of the plane the function $f(x, y) = y^3 - 3yx^2$ is always positive or always negative. All we need to do is to evaluate f at one point of each region. The values are alternately positive and negative. Hence the surface M is alternately above and below the plane. This surface is illustrated in Figure 5.22 on page 205.

IV.2 PATCH COMPUTATIONS

Introduction

This section follows on directly from Section IV.1 and assumes familiarity with curves, dealt with in Section I.4, and the dot and cross product, dealt with in Section II.1. It is also useful to bear in mind the definition of the derivative of a mapping, given in Section I.7.

Choosing patches enables us to describe surfaces in terms of regions of the plane E^2. In this section we begin to see how we can use the geometry of E^2 to study surfaces.

This section contains many important examples, which will be used again and again throughout the remainder of the course.

READ: Section IV.2 (pages 133 - 140).

Comment

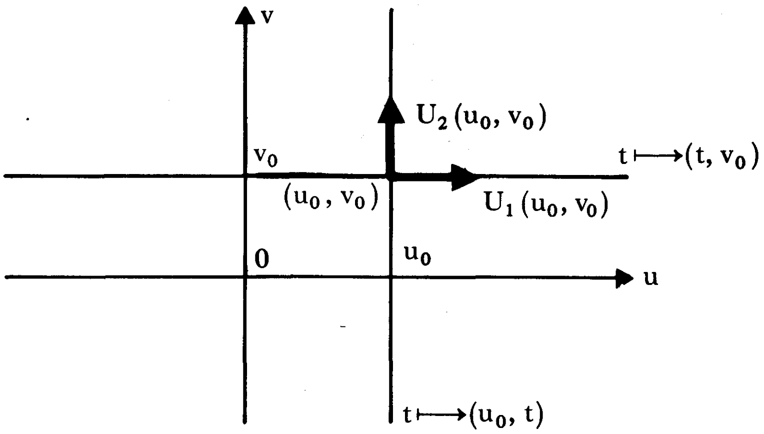
(i) *Patches* Through each point (u_0, v_0) in the plane E^2 there are two curves

$$t \longmapsto (t, v_0)$$

and $t \longmapsto (u_0, t)$,

that are parallel to the coordinate axes.

These are unit-speed curves and their velocities at the point (u_0, v_0) are $U_1(u_0, v_0)$ and $U_2(u_0, v_0)$.



The images of these curves under the mapping x are the u- and v-parameter curves, $v = v_0$ and $u = u_0$ respectively. These parameter curves depend on the choice of coordinate patch x.

Since regular mappings preserve velocity vectors, the partial velocities (the velocities of the parameter curves) are given by

$$x_u(u_0, v_0) = x_*(U_1(u_0, v_0))$$

and

$$x_v(u_0, v_0) = x_*(U_2(u_0, v_0)).$$

Hence $\mathbf{x}_u$ and $\mathbf{x}_v$ are the composite mappings $\mathbf{x}_*(U_1)$ and $\mathbf{x}_*(U_2)$. To remind ourselves that $\mathbf{x}_u$ and $\mathbf{x}_v$ are functions on E^2 we sometimes write them as $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$. To calculate these explictly we look at the parameter curves. The u-parameter curve $v = v_0$ is given by

$$\alpha : t \longmapsto \mathbf{x}(t, v_0) = (x_1(t, v_0), x_2(t, v_0), x_3(t, v_0)).$$

Hence

$$\mathbf{x}_u(u_0, v_0) = \frac{d\alpha}{dt}(u_0) \quad \text{at } \mathbf{x}(u_0, v_0)$$

$$= (\frac{\partial x_1}{\partial u}(u_0, v_0), \frac{\partial x_2}{\partial u}(u_0, v_0), \frac{\partial x_3}{\partial u}(u_0, v_0)) \quad \text{at } \mathbf{x}(u_0, v_0).$$

Since $\mathbf{x}$ is regular, the derivate $\mathbf{x}_*$ is one-to-one and hence it carries linearly independent vectors into linearly independent vectors. The tangent vectors $U_1(u_0, v_0)$ and $U_2(u_0, v_0)$ are linearly independent in E^2 and so $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$ are linearly independent in E^3. In Section II.1 we saw that this is equivalent to the condition

$$\mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0) \neq \mathbf{0}.$$

That is, $\mathbf{x}_u \times \mathbf{x}_v$ is never zero.

In fact this condition that $\mathbf{x}_u \times \mathbf{x}_v$ is never zero is enough to ensure that $\mathbf{x}$ is regular. The condition ensures that the images of $U_1(u_0, v_0)$ and $U_2(u_0, v_0)$ under $\mathbf{x}_*$ are linearly independent and this implies that $\mathbf{x}_*$ is one-to-one at each point.

When O'Neill comes to calculate $\mathbf{x}_u \times \mathbf{x}_v$ he obtains the equation

$$\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix}$$

Here U_1, U_2, U_3 are vector valued functions with domain E^3 while all the other functions have domain E^2 and hence the equation is not technically correct. However it is very convenient to use such expressions as long as you are always aware of how they could be made rigorous. To make it rigorous we could replace U_1, U_2, U_3 by the composite functions $U_1 \circ \mathbf{x}$, $U_2 \circ \mathbf{x}$ and $U_3 \circ \mathbf{x}$, or state that if the functions with the domain E^2 are evaluated at the point (u_0, v_0) then those with domain E^3 are evaluated at the point $\mathbf{p} = \mathbf{x}(u_0, v_0)$.

Supplementary Comments

(i) *Page 136: line -1*

$$r^{-2}\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ -\cos v \sin u & \cos v \cos u & 0 \\ -\sin v \cos u & -\sin v \sin u & \cos v \end{vmatrix}$$

$$= \cos^2 v \cos u \, U_1 - \cos v\,(-\cos v \sin u)\, U_2$$

$$+((-\cos v \sin u)(-\sin v \sin u) - (-\sin v \cos u)(\cos v \cos u))\, U_3$$

$$= \cos^2 v \cos u \, U_1 + \cos^2 v \sin u \, U_2 + \cos v \sin v \, U_3$$

since $\cos^2 u + \sin^2 u = 1$.

(ii) *Page 139: line 2, $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent*

Here

$$\mathbf{x}_u(u, v) = (g'(u), h'(u) \cos v, h'(u) \sin v)$$
$$\mathbf{x}_v(u, v) = (0, -h(u) \sin v, h(u) \cos v)$$

and hence

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = (h(u)\, h'(u), -h(u)g'(u) \cos v, -h(u)g'(u) \sin v).$$

If $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$ are linearly dependent at $\mathbf{x}(u_0, v_0)$ then

$$\mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0) = 0.$$

But since $h(u_0) > 0$ and $\cos v_0$ and $\sin v_0$ cannot simultaneously be zero, we would have $h'(u_0) = g'(u_0) = 0$. This is impossible since by the definition (on page 21) a parametrization is a regular curve. Hence $\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)$ is never zero and the partial velocities $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$ are linearly independent.

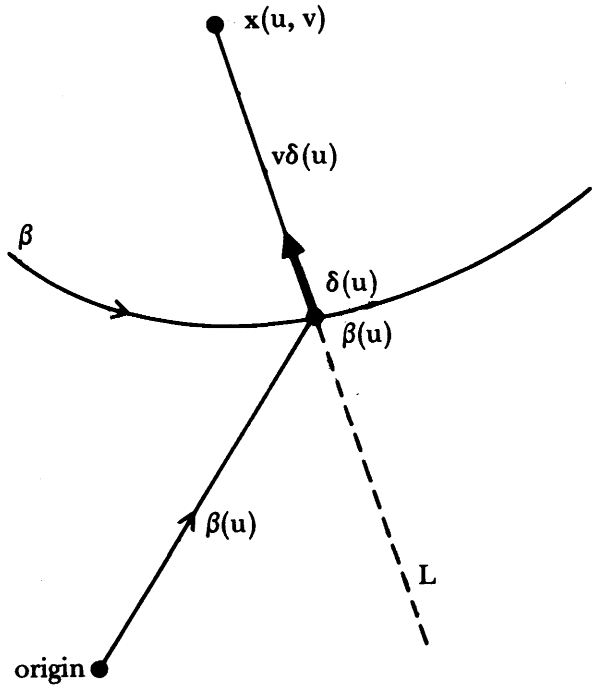
Additional Text

There are two important classes of surfaces defined in the Exercises at the end of Section IV.2.

(i) *Ruled Surfaces*

READ the definition on page 140 between Exercises 3 and 4 and look at the examples in Exercises 5 and 6.

Comment $\beta(u)$ gives the position along the base curve, $\delta(u)$ gives the 'direction' of the line L and v gives the 'distance' along this line from the base curve.



δ(u) need not be the same length and direction for each u.

(ii) *Quadric surfaces*

READ the definition on page 142, between Exercises 9 and 10, and browse through the examples in Exercises 10 to 11.

Summary

Notation

$x_u(u_0,v_0)$ and $x_v(u_0,v_0)$	Page 134, Definition 2.1
x_u and x_v	Page 134, line 12

Definitions

(i)	u- and v- parameter curves, $v = v_0$ and $u = u_0$	Page 133, line -9
(ii)	Partial velocities, $x_u(u_0,v_0)$ and $x_v(u_0,v_0)$	Page 134, Definition 2.1
(iii)	Parametrization	Page 135, Definition 2.3
(iv)	Cross-sectional curve of a cylinder	Page 138, line 9
(v)	Rulings of a cylinder	Page 138, line 11
(vi)	Ruled surface	Page 140, lines -16 to -8
(vii)	Rulings	
(viii)	Ruled form	
(ix)	Base curve	
(x)	Director curve	
(xi)	Quadric surface	Page 142, line 16

Examples

(i)	The geographical patch on the sphere: $\mathbf{x}(u, v) = (r \cos v \cos u, r \cos v \sin u, r \sin v)$, where $u \in (-\pi, \pi), v \in (-\frac{\pi}{2}, \frac{\pi}{2})$	Page 135, line 3
(ii)	Parametrization of a cylinder: $\mathbf{x}(u, v) = (\alpha_1 (u), \alpha_2 (u), v)$	Page 137, Example 2.4
(iii)	Parametrization of a surface of revolution: $\mathbf{x}(u, v) = (g(u), h(u) \cos v, h(u) \sin v)$	Page 138, Example 2.5
(iv)	Parametrization of a torus of revolution: $\mathbf{x}(u, v) = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u)$	Page 139, Example 2.6
(v)	Parametrization of a ruled surface in ruled form: $\mathbf{x}(u, v) = \beta(u) + v\delta(u)$	Page 140, line -12
(vi)	Helicoid: $\mathbf{x}(u, v) = (u \cos v, u \sin v, bv)$	Page 141, Exercise 7

Results

(i)	$\mathbf{x} : D \longrightarrow E^3$ is regular if and only if $\mathbf{x}_u \times \mathbf{x}_v$ is never zero.	Page 136, line 6
(ii)	The partial velocities are the images of the standard basis vectors under the derivative mapping $\mathbf{x}_*$: $\mathbf{x}_*(U_1) = \mathbf{x}_u, \quad \mathbf{x}_*(U_2) = \mathbf{x}_v$.	Text, page 20

Techniques

(i)	Description of the parameter curves.	Page 133, line -9
(ii)	Evaluation of partial velocities.	Page 134, line -7
(iii)	Testing regularity of a parametrization.	Page 136, line 6
(iv)	Use of standard parametrization of cylinders and surfaces of revolution.	Page 137, Example 2.4, Page 138, Example 2.5
(v)	Finding the ruled form of a surface, in certain simple examples.	Page 140, line -16
(vi)	Familiarity with the shapes and equations of quadric surfaces.	Page 142, line 16

Exercises

Techniques (i), (ii) and (iii)

1. Page 141, Exercise 7.

Technique (iii)

2. Page 140, Exercise 2.

Technique (iv)

3. Page 141, Exercise 6.
4. Page 140, Exercise 1.

Technique (v)

5. Page 140, Exercise 4.

Technique (vi)

6. Page 142, Exercises 10 and 11. Only sketch the surfaces.
7. Page 142, Exercise 12(a) and (b).

Solutions

1. Page 141, Exercise 7.

(a) To check that **x** is a patch we must show that it is one-to-one and regular. If $\mathbf{x}(u, v) = \mathbf{x}(u', v')$ then

$$u \cos v = u' \cos v', \quad u \sin v = u' \sin v' \quad \text{and} \quad v = v'.$$

Since $v = v'$ and the cosine and sine functions cannot be simultaneously zero it follows that $u = u'$. Hence **x** is one-to-one.

Now

$$\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & b \end{vmatrix}$$

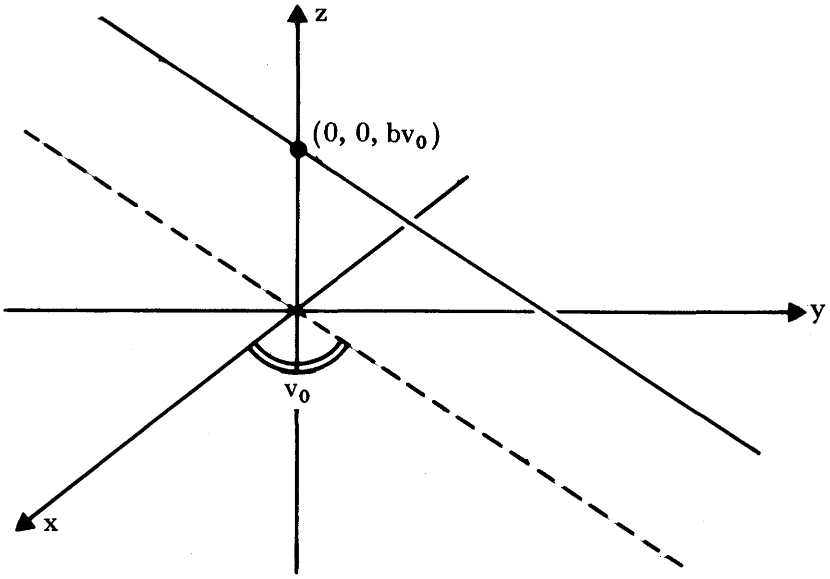
$$= (b \sin v, - b \cos v, u).$$

Hence $\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = b^2 + u^2 > 0$, and so $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$. Thus **x** is regular and therefore a patch.

(b) The u-parameter curve $v = v_0$ is

$$\begin{aligned} t \longmapsto \mathbf{x}(t, v_0) &= (t \cos v_0, t \sin v_0, bv_0) \\ &= (0, 0, bv_0) + t(\cos v_0, \sin v_0, 0), \end{aligned}$$

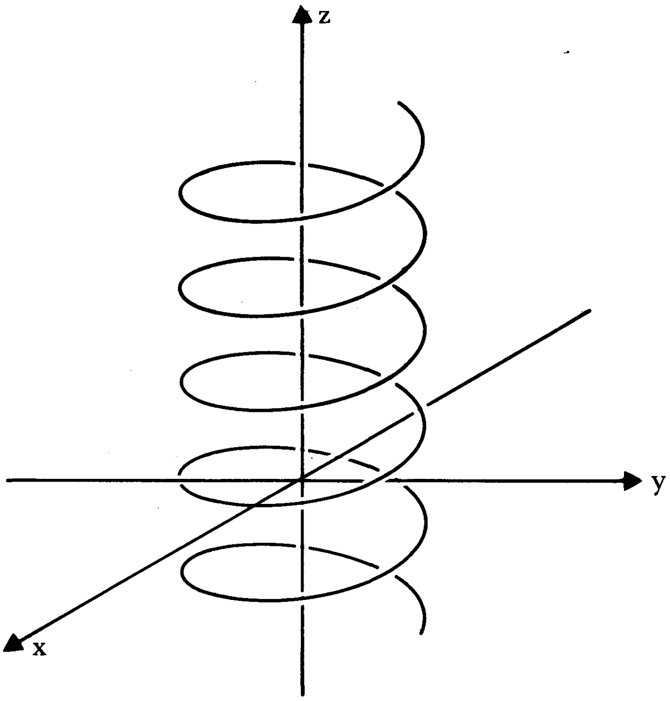
which is a straight line parallel to the line in the xy plane making an angle v_0 with the x axis.



The v-parameter curve $u = u_0$ is

$$t \longmapsto \mathbf{x}(u_0, t) = (u_0 \cos t, u_0 \sin t, bt).$$

As t increases by 2π the curve goes once round the z axis, in a circle of radius u_0, and rises $2\pi b$. The v-parameter curves are helices about the z axis.



(c) In order to express the helicoid in implicit form we must find some function g and constant c such that $g(\mathbf{p}) = c$ if and only if $\mathbf{p} = \mathbf{x}(u_0, v_0)$ for some (u_0, v_0).

If $\mathbf{p} = \mathbf{x}(u_0, v_0)$ then

$$p_3 = bv_0, \quad v_0 = p_3/b$$

and

$$p_1 = u_0 \cos(p_3/b), \quad p_2 = u_0 \sin(p_3/b).$$

Hence for each point p of the helicoid

$$p_1 \sin(p_3/b) = p_2 \cos(p_3/b)$$

so if a point belongs to the helicoid it also belongs to the implicitly defined surface

$$M : x \sin(z/b) - y \cos(z/b) = 0.$$

We need to show conversely that any point of M belongs to the helicoid. Suppose p satisfies the above relationship. Then since $b \neq 0$ we can set $v_0 = p_3/b$ and we obtain

$$p_1 \sin v_0 = p_2 \cos v_0.$$

For any value of v_0 at least one of $\cos v_0$, $\sin v_0$ must be nonzero. Suppose $\cos v_0 \neq 0$. Then we can find $u_0 = p_1/\cos v_0$. Now $p_2 \cos v_0 = p_1 \sin v_0 = u_0 \cos v_0 \sin v_0$ and so, since $\cos v_0 \neq 0$, $p_2 = u_0 \sin v_0$. Hence we can write p in the form $(u_0 \cos v_0, u_0 \sin v_0, bv_0)$ and so it belongs to the helicoid.

A similar argument works when $\sin v_0 \neq 0$.

Hence the helicoid is the surface

$$M : x \sin(z/b) - y \cos(z/b) = 0.$$

2. Page 140, Exercise 2.

By Lemma II.1.8,

$$\begin{aligned} \|\mathbf{x}_u \times \mathbf{x}_v\|^2 &= (\mathbf{x}_u \cdot \mathbf{x}_u)(\mathbf{x}_v \cdot \mathbf{x}_v) - (\mathbf{x}_u \cdot \mathbf{x}_v)^2 \\ &= EG - F^2. \end{aligned}$$

Now $\mathbf{x}$ is regular if and only if $\mathbf{x}_u \times \mathbf{x}_v \neq 0$. But $\mathbf{x}_u \times \mathbf{x}_v$ is never zero if and only if $\|\mathbf{x}_u \times \mathbf{x}_v\|$ is never zero, which is true if and only if $EG - F^2$ is never zero.

3. Page 141, Exercise 6.

$$\mathbf{x}(u, v) = \beta(u) + v\mathbf{q}.$$

Hence $\mathbf{x}_u = \beta'$

and $\mathbf{x}_v = \mathbf{q}$:

$$\mathbf{x}_u \times \mathbf{u}_v = \beta' \times \mathbf{q},$$

and so $\mathbf{x}$ is regular if and only if $\beta' \times \mathbf{q}$ is never zero.

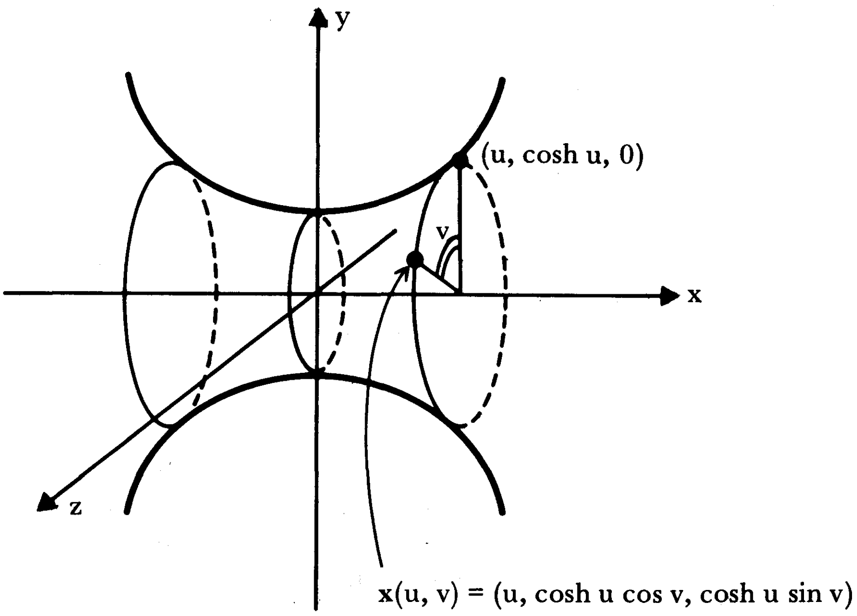
In Example 2.4 the curve β is in the xy plane and $\mathbf{q} = (0, 0, 1)$.

4. Page 140, Exercise 1.

(a) A parametrization of the profile curve is $u \longmapsto (u, \cosh u, 0)$.
A parametrization of the catenoid is

$$\mathbf{x} : (u, v) \longmapsto (u, \cosh u \cos v, \cosh u \sin v)$$

since rotation about the x axis keeps the x coordinate fixed.



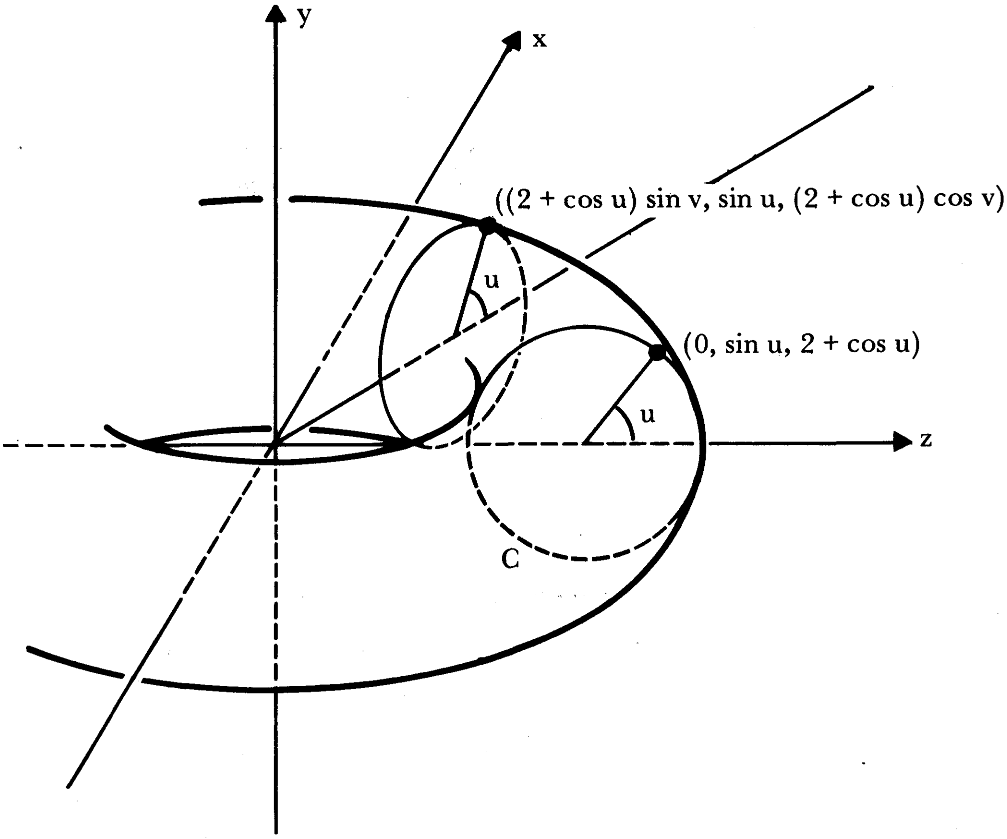
(b) C : $(z - 2)^2 + y^2 = 1$ is a circle in the yz plane of radius one and centre (0, 2). A possible parametrization is

$$u \longmapsto (0, \sin u, 2 + \cos u).$$

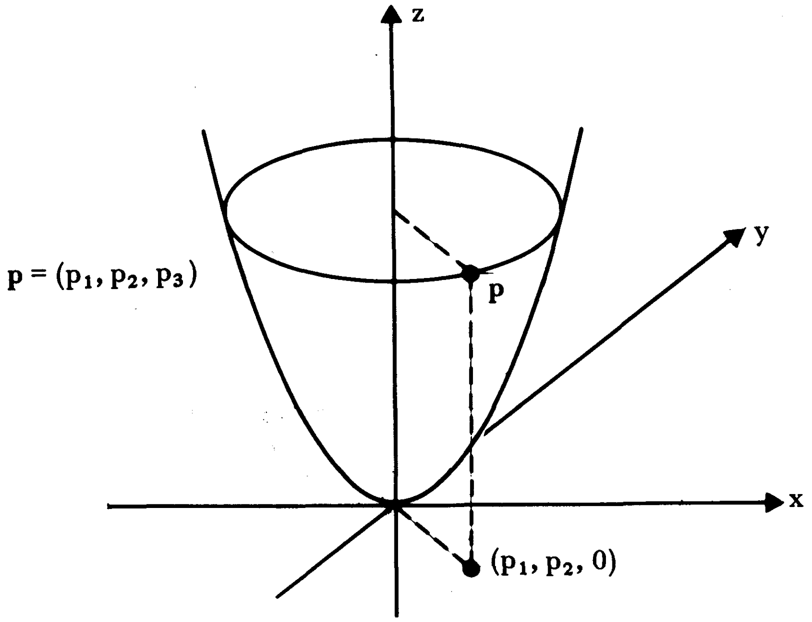
A parametrization of the torus is

$$x : (u, v) \longmapsto ((2 + \cos u) \sin v, \sin u, (2 + \cos u) \cos v),$$

since rotation about the y axis keeps the y coordinate fixed.



(c) We cannot follow the same procedure as in parts (a) and (b) since the profile curve cuts the axis of rotation and hence the standard 'parametrization' will not be regular. However we can describe the surface in terms of a Monge patch.



A point is on the paraboloid M if the square of its distance from the z axis is equal to its height above the xy plane. Thus $p = (p_1, p_2, p_3)$ belongs to M if and only if $p_1^2 + p_2^2 = p_3$. Hence $M : z = x^2 + y^2$ and we can parametrize it by mapping each point in the xy plane to the unique point on the surface above it. This gives the single patch

$$\mathbf{x} : (u, v) \longmapsto (u, v, u^2 + v^2).$$

5. Page 140, Exercise 4.

Look at the intersection of M with the plane $x = c$, for some constant c. This is the straight line $z = cy$ in the plane $x = c$. It passes through the points $(c, 0, 0)$ and $(c, 1, c)$. We can write it as

$$v \longmapsto (c, 0, 0) + v(0, 1, c).$$

Letting the point $(c, 0, 0)$ vary along the x axis we obtain the parametrization

$$(u, v) \longmapsto (u, 0, 0) + v(0, 1, u),$$

which is in ruled form.

Similarly looking at the intersection of M with planes of the form $y = c$ we find the parametrization

$$(u, v) \longmapsto (0, v, 0) + u(1, 0, v).$$

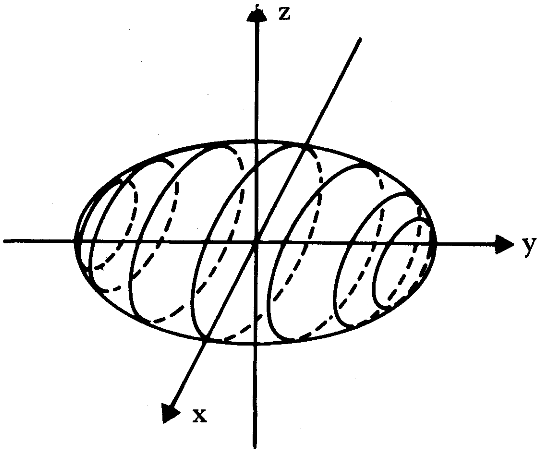
Alternatively, since $M : z = xy$, M is described by the Monge patch

$$\mathbf{x} : (u, v) \longmapsto (u, v, uv),$$

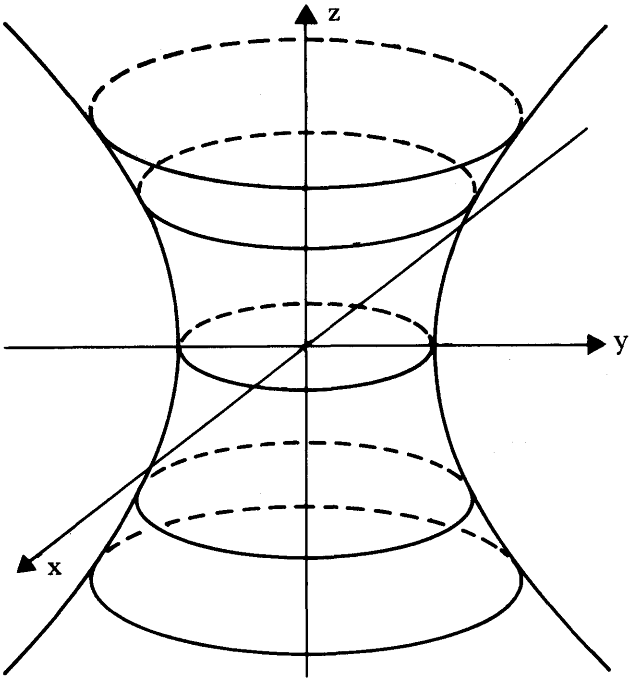
from which the above two ruled forms follow.

6. Page 142, Exercises 10 and 11.

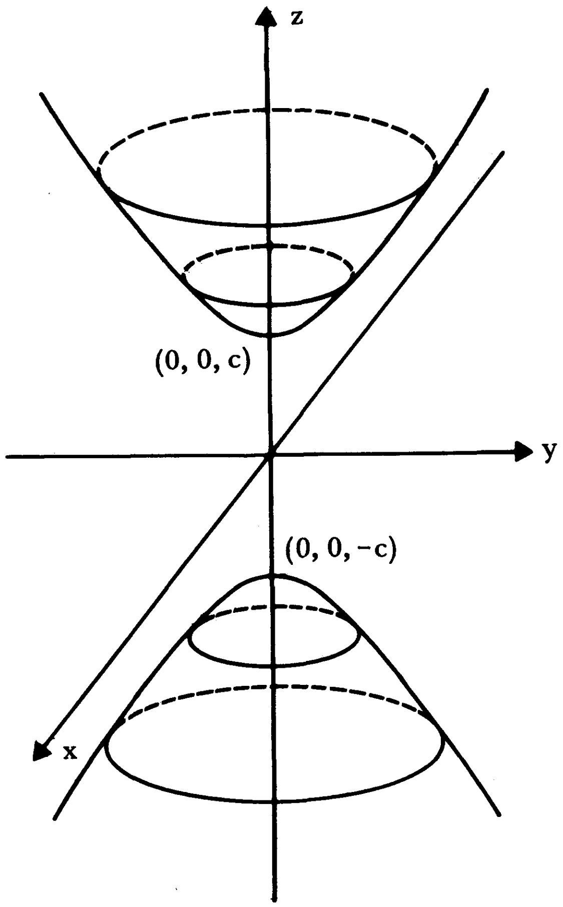
10(a). Ellipsoid



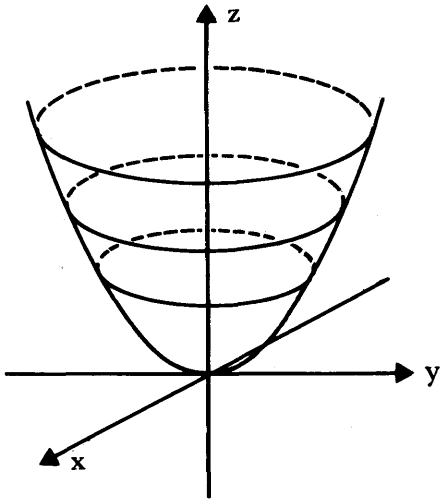
10(b). Elliptic hyperboloid



10(c). Elliptic hyperboloid (two sheets)



11. Elliptic paraboloid



7. Page 142, Exercise 12.

(a) The mapping $\mathbf{x} : E^2 \longrightarrow E^3$ does map into M since

$$4uv = \frac{(a(u+v))^2}{a^2} - \frac{(b(u-v))^2}{b^2}.$$

We need to show further that it maps onto M. For any point **p** belonging to M we need to solve the equations

$$p_1 = a(u_0 + v_0)$$

$$p_2 = b(u_0 - v_0)$$

$$p_3 = 4u_0 v_0.$$

We can solve the first two, obtaining

$$u_0 = \frac{p_1}{2a} + \frac{p_2}{2b}, \quad v_0 = \frac{p_1}{2a} - \frac{p_2}{2b}$$

and since **p** belongs to M the third equation is also satisfied.

Next we need to check that **x** is one-to-one. Suppose

$$\mathbf{x}(u, v) = \mathbf{x}(u', v'):$$

then

$$u + v = u' + v'$$

$$u - v = u' - v'$$

and hence $u = u'$, $v = v'$. The mapping **x** is one-to-one.

Now we need to check that **x** is regular: we use the usual method.

$$\mathbf{x}_u \times \mathbf{x}_u = \begin{vmatrix} U_1 & U_2 & U_3 \\ a & b & 4v \\ a & -b & 4u \end{vmatrix}$$

$$= (4b(u+v), 4a(v-u), -2ab).$$

Now, for the definition of the hyperbolic paraboloid to be meaningful we need $a,b > 0$. Hence $-2ab \neq 0$ and so $\mathbf{x}_u \times \mathbf{x}_v$ is never zero. This in turn proves that $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent and hence **x** is regular.

Thus **x** is indeed a patch and to complete the proof we need only prove that **x** is proper. To prove that **x** is proper we have to show that the inverse function $\mathbf{x}^{-1} : M \longmapsto E^2$ is continuous. Since $\mathbf{x}^{-1}$ is given by the formula

$$\left(\frac{x}{2a} + \frac{y}{2b}, \frac{x}{2a} - \frac{y}{2b}\right)$$

continuity follows from standard results in analysis.

(b) We can write **x** in ruled form as follows

(i) $\mathbf{x}(u, v) = (au, bu, 0) + v(a, -b, 4u)$

and

(ii) $\mathbf{x}(u, v) = (av, -bv, 0) + u(a, b, 4v).$

IV.3 DIFFERENTIABLE FUNCTIONS AND TANGENT VECTORS

Introduction

This section follows on from the previous section, IV.2, and follows the development of Chapter I, Sections 1 to 4. You will also need the chain rule dealt with in Section I.7.

The section begins with certain rather technical but intuitively obvious definitions and results. Mappings that you would expect to be differentiable turn out to be differentiable, and, given a coordinate patch, we can use it to give explicit descriptions of curves in a surface. Do not worry about the technicalities.

We then show that the partial velocities, for a given patch, do tell us something independent of the choice of patch. The tangents to any curve in the surface can be written as a linear combination of them. These tangents form the ***tangent planes*** to the surface. We use them to define a ***directional derivative*** for functions defined only on the surface, and not on the whole of E^3.

READ: Section IV.3 (pages 143 - 149). Do not spend much time on pages 143 - 145. Just try to understand the results.

Comments

(i) *Differentiation on M and on E^3* We have now met several different definitions of differentiation with domain or codomain either E^3 or some surface M. If we are dealing with a function $F : E^n \longrightarrow E^m$ between Euclidean spaces and we are using the definition of differentiability given in Section I.7 we say that F is Euclidean differentiable. If we are using either of the definitions of this section for functions with domain or codomain some surface M we indicate it by saying that $G : M \longrightarrow E^n$ is differentiable or $F : E^n \longrightarrow M$ is differentiable. We need to compare these definitions of differentiability with the Euclidean definition.

Directly from the definitions we have that $G : M \longrightarrow E^n$ or $F : E^n \longrightarrow M$ are differentiable if their respective coordinate expressions are Euclidean differentiable.

Suppose we have a function $F : E^n \longrightarrow M$. Then we can also consider it as a function $F : E^n \longrightarrow E^3$. If we know that F is differentiable as a function from E^n to M then we would hope that F is also Euclidean differentiable. It is. Since $F : E^n \longrightarrow M$ is differentiable it follows, by definition, that $x^{-1} \circ F : E^n \longrightarrow E^2$ is Euclidean differentiable, for each patch x. But by the definition of a patch, each patch $x : D \longrightarrow E^3$ is Euclidean differentiable. Consequently the composite mappings $x \circ x^{-1} \circ F : E^n \longrightarrow E^3$ are Euclidean differentiable. But each of these mappings is just the restriction of F to those points whose image under F is in x(D). Since the patches cover M we can choose enough "x"s to deal with F at every point. Hence $F : E^n \longrightarrow E^3$ is Euclidean differentiable.

Theorem 3.2 gives us the converse of this result. If $F : E^n \longrightarrow E^3$ is a Euclidean differentiable mapping whose image lies in M then it is also differentiable considered as a function $F : E^n \longrightarrow M$.

The corollaries are proved as follows.

Corollary 3.3 Since y is a patch it is a differentiable function $y : D \longrightarrow E^3$ whose image lies in M. Hence, by Theorem 3.2, it is also differentiable considered as a function $y : D \longrightarrow M$. By definition this just means that $x^{-1} \circ y$ is a Euclidean differentiable mapping between suitable subsets of E^2.

Corollary 3.4 Since $x^{-1}y$ is Euclidean differentiable

$$x^{-1}y(u, v) = (\bar{u}(u, v), \bar{v}(u, v)),$$

where the Euclidean coordinate functions $\bar{u}$, $\bar{v}$ are differentiable. Applying the mapping x to each side gives the result. The mapping $x^{-1}\,y = (\bar{u}, \bar{v})$ gives us a coordinate transformation.

Theorem 3.2 can also be used to give a converse of *Lemma 3.1.* Suppose a function $\alpha : I \longrightarrow M$ is such that

$$\alpha(t) = x(a_1(t),\ a_2(t))$$

for differentiable functions a_1 and a_2. Then the composite $x(a_1, a_2)$ is a Euclidean differentiable mapping into E^3, whose image lies in M. Hence by Theorem 3.2 its restriction $\alpha : I \longrightarrow M$ is differentiable; that is, α is a curve.

As a corollary to this we find that the parameter 'curves' are in fact curves in M since they can be written as

$$t \longmapsto x(t, v_0)$$

$$t \longmapsto x(u_0, t),$$

where the functions given by $t \longmapsto t$, $t \longmapsto v_0$, $t \longmapsto u_0$ are all differentiable.

When no confusion can arise we omit the word 'Euclidean' from the expression 'Euclidean differentiable'.

(ii) *Page 147: line 1* We have two mappings F and x with the following Euclidean coordinate functions:

$$F = (a_1, a_2) \quad \text{and} \quad x = (x_1, x_2, x_3).$$

The composite mapping $x \circ F$ is α. The composite (chain) rule gives

$$\alpha_* = x_*(F) \circ F_*.$$

In terms of Jacobian matrices this becomes

$$\begin{pmatrix} \alpha_1' \\ \\ \alpha_2' \\ \\ \alpha_3' \end{pmatrix}_t = \begin{pmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_1}{\partial v} \\ \\ \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_2}{\partial v} \\ \\ \dfrac{\partial x_3}{\partial u} & \dfrac{\partial x_3}{\partial v} \end{pmatrix}_{F(t)} \begin{pmatrix} a_1' \\ \\ a_2' \end{pmatrix}_t$$

Multiplying out we obtain

$$\begin{aligned} \alpha'(t) &= x_u(F(t))a_1'(t) + x_v(F(t))a_2'(t) \\ &= x_u(a_1(t), a_2(t))a_1'(t) + x_v(a_1(t), a_2(t))a_2'(t). \end{aligned}$$

(iii) *Page 147: vector fields* Since $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$ are tangent vectors at $x(u_0, v_0)$ we can consider x_u and x_v as tangent vector fields on $x(D)$. Since $x_u \times x_v$ is orthogonal to both x_u and x_v we can consider it as a nonvanishing normal vector field on $x(D)$. It may be impossible to extend this to a nonvanishing normal vector field on the whole of M.

(iv) *Page 148: the proof of Lemma 3.8* It is possible to have implicitly defined subsets $M : g = c$ in E^3 that are surfaces but for which dg and hence the derivatives $\partial g/\partial x_i$ are zero. An example of such a subset is $M : x^2 = 0$. The only nontrivial derivative is $\partial g/\partial x = 2x$, which is identically zero on M. Hence the converse of Theorem IV.1.4 is false. In this lemma we need to add the restriction that dg is nonzero on M.

The form of the composite (chain) rule used is as follows. We have

$$\alpha : t \longmapsto (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$

$$g : (p_1, p_2, p_3) \longmapsto g(p),$$

and $g\alpha : t \longmapsto c$ (constant).

Hence

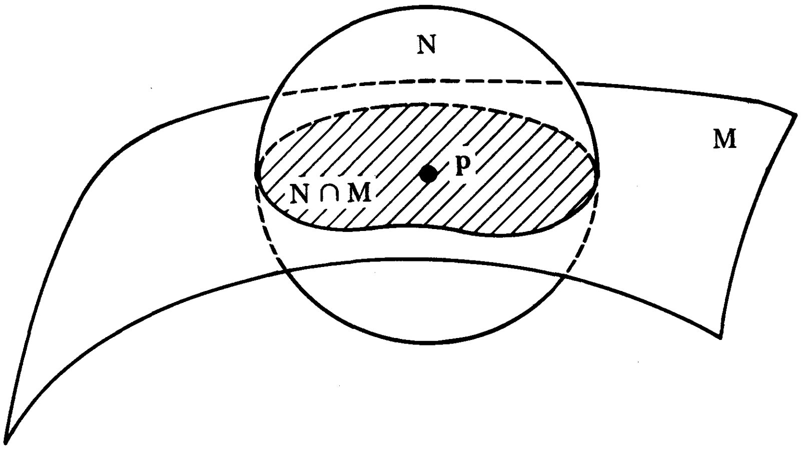
$$0 = (g\alpha)'(t) = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}\right)_{\alpha(t)} \begin{pmatrix} \alpha_1' \\ \alpha_2' \\ \alpha_3' \end{pmatrix}_t$$

which gives the required result.

(v) *Page 149: Definition 3.10* We have defined the directional derivative, $v[f]$, of a function f on M as the common value of $(d(f\alpha)/dt)(0)$ for all curves α in M with initial velocity v.

It is not at all obvious that this directional derivative is well defined. Given any vector v tangent to M we first of all have to assure ourselves that there is a curve α with initial velocity v and then we have to assure ourselves that the value of $(d(f\alpha)/dt)(0)$ is independent of which curve α we choose. The first point is easily settled. Since v is a tangent to M, by Definition 3.5 it is the velocity vector of some curve in M and by suitably shifting the parametrization of the curve we can ensure that it is the initial velocity of a curve α in M.

The second point is rather more complicated. To apply Theorem I.4.6 we need to extend f to a differentiable function F defined on some neighbourhood N of p in E^3.



If q is in the intersection of N and M then $f(q) = F(q)$.

It is always possible to extend the function f in this way and then we can apply results on directional derivatives in E^3 to the function F. Of course, we need to be even more careful since it is possible to choose F in many distinct ways. At this stage we can use Theorem I.4.6. It tells us that

$$v[F] = \beta'(0)[F] = \frac{d(F\beta)}{dt}(0),$$

for any curve β with initial velocity v. Hence

$$v[F] = \frac{d(F\alpha)}{dt}(0)$$

for any curve α in M with initial velocity v. Now since F is an extension of f we find that

$$v[F] = \frac{d(f\alpha)}{dt}(0)$$

for any curve α in M with initial velocity v. Looking at the left hand expression we see that it does not mention any curve α and so is independent of that choice. Looking at the right hand expression we see that it does not mention any extension F and so is independent of that choice. Since the left hand expression equals the right hand expression and one is independent of the choice of α and the other is independent of the choice of F it follows that they are both independent of the choice of either α or F. Hence the expressions $(d(f\alpha)/dt)\ (0)$ do have a common value and we can define

$$v[f] = \frac{d(f\alpha)}{dt}(0).$$

This argument also suggests an alternative method of evaluating v[f]. We have seen that

$$v[f] = \frac{d(f\alpha)}{dt}(0) = \frac{d(F\alpha)}{dt}(0) = v[F]$$

for any suitable extension F. Hence if we can extend f, and in most cases this is automatic, we can evaluate v[f] as v[F] and use all the techniques developed in Section I.3.

It follows that this new directional derivative satisfies the usual linear and Leibnizian properties (Theorem I.3.3) when we restrict attention to functions on M and vectors tangent to M.

We shall need to find directional derivatives with respect to the partial velocities $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$. Now $x_u(u_0, v_0)$ is the initial velocity of the curve $\alpha(t) = x(u_0 + t, v_0)$ and so

$$x_u(u_0, v_0)[f] = \frac{d(fx(u_0 + t, v_0))}{dt}(0) = \frac{\partial(fx)}{\partial u}(u_0, v_0).$$

Similarly

$$x_v(u_0, v_0)[f] = \frac{\partial(fx)}{\partial v}(u_0, v_0).$$

Letting the point (u_0, v_0) vary throughout the domain of the patch $\mathbf{x}$ we find that

$$\mathbf{x}_u[f] = \frac{\partial(f\mathbf{x})}{\partial u} \quad \text{and} \quad \mathbf{x}_v[f] = \frac{\partial(f\mathbf{x})}{\partial v}.$$

In the usual pointwise manner we can extend the definition of the directional derivatives to apply to any tangent vector field V on M. For any real differentiable function f on M we define a new differentiable function V[f] as follows:

$$V[f](\mathbf{p}) = V(\mathbf{p})[f],$$

for any point $\mathbf{p}$ in M.

Summary

Notation

$\left.\begin{array}{l}\alpha(t) = \mathbf{x}(a_1(t), a_2(t))\\ \alpha = \mathbf{x}(a_1, a_2)\end{array}\right\}$	Page 144, Lemma 3.1
$T_p(M)$	Page 146, line 15
∇g	Page 148, Lemma 3.8
$v[f]$	Page 149, Definition 3.10

Definitions

(i)	Coordinate expression $f(\mathbf{x})$ or $\mathbf{x}^{-1}(F)$	Page 143, line −9 and Page 143, line −5
(ii)	Differentiable functions on or to M	Page 143, line 8 and Page 143, line −3
(iii)	Curve in a surface	Page 144, line 3
(iv)	Coordinate expressions for a curve in a surface: $\alpha(t) = \mathbf{x}(a_1(t), a_2(t))$ or $\alpha = \mathbf{x}(a_1, a_2)$	Page 144, Lemma 3.1
(v)	Coordinate transformation, $\mathbf{x}^{-1}\mathbf{y} = (\bar{u}, \bar{v})$	Text, page 34, line 9
(vi)	Tangent to M at $\mathbf{p}$	Page 146, Definition 3.5
(vii)	Tangent plane $T_p(M)$	Page 146, line 14
(viii)	Euclidean vector field on M	Page 147, Definition 3.7
(ix)	Tangent vector field on M	Page 147, line −10
(x)	Vector normal to M	Page 147, line −6
(xi)	Normal vector field on M	Page 147, line −4
(xii)	Gradient ∇g	Page 148, Lemma 3.8
(xiii)	Directional derivative $(v[f])$ of a function on M	Page 149, Definition 3.10

Example

(i)	$\Sigma x_i U_i$ is a normal vector field on the sphere $\Sigma : g = \Sigma x_i^2 = r^2$.	Page 148, Example 3.9

Results

(i)	Any curve α in M whose image lies in $\mathbf{x}(D)$ can be expressed as $\alpha(t) = \mathbf{x}(a_1(t), a_2(t))$.	Page 144, Lemma 3.1
(ii)	If $F : E^n \longrightarrow E^3$ is differentiable and its image is in M then $F : E^n \longrightarrow M$ is differentiable.	Page 145, Theorem 3.2
(iii)	If $\mathbf{x}$, $\mathbf{y}$ are patches then $\mathbf{x}^{-1}\mathbf{y}$ and $\mathbf{y}^{-1}\mathbf{x}$ are differentiable.	Page 145, Corollary 3.3
(iv)	If $\mathbf{x}$, $\mathbf{y}$ are overlapping patches there are differentiable functions $(\bar{u}, \bar{v})$ such that $\mathbf{y}(u, v) = \mathbf{x}(\bar{u}(u, v), \bar{v}(u, v))$.	Page 145, Corollary 3.4
(v)	$\mathbf{v}$ is a tangent vector to M at $\mathbf{p} = \mathbf{x}(u_0, v_0)$ if and only if $\mathbf{v}$ is a linear combination of $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$.	Page 146, Lemma 3.6
(vi)	If α is a curve in M such that $\alpha = \mathbf{x}(a_1, a_2)$, then $\alpha' = \mathbf{x}_u(a_1, a_2)a_1' + \mathbf{x}_v(a_1, a_2)a_2'$.	Page 147, line 1
(vii)	If $M : g = c$ is a surface and dg is never zero on M then the gradient ∇g is a nonvanishing normal vector field.	Page 148, Lemma 3.8
(viii)	$\mathbf{x}_u \times \mathbf{x}_v$ is a normal vector field on $\mathbf{x}(D)$.	Text, page 35, line 3
(ix)	For any patch $\mathbf{x}$: $\mathbf{x}_u[f] = \frac{\partial}{\partial u}(f\mathbf{x})$ and $\mathbf{x}_v[f] = \frac{\partial}{\partial v}(f\mathbf{x})$.	Text, page 37, line 3

Techniques

(i)	Deciding whether a function to or from M is differentiable by finding its coordinate expressions.	Page 143, line −9, Page 143, line −5, and Page 144, Lemma 3.1
(ii)	Finding coordinate transformations between patches.	Page 145, Corollary 3.4
(iii)	Determining whether a vector is a tangent to M at $\mathbf{p} = \mathbf{x}(u_0, v_0)$ by checking whether it is a linear combination of $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$.	Page 146, Lemma 3.6
(iv)	Finding normal vector fields on M, $\mathbf{x}_u \times \mathbf{x}_v$ or ∇g, and using them to find vectors tangent to M.	Page 148, Lemma 3.8 and Text, page 35, line 3
(v)	Evaluating the directional derivatives of a function on a surface either from first principles, or by using the linear and Leibnizian properties, or by extending the function to some neighbourhood of the surface in E^3 and then applying standard techniques, or by using the formulas of Result (ix)	Page 149, Definition 3.10, Text, page 36, and Text, page 37

Exercises

Technique (i)

1. Page 149, Exercise 1.
2. Page 149, Exercise 2.

Technique (ii)

3. Page 150, Exercise 6.

Technique (iii)

4. Page 150, Exercise 8(a).

Technique (iv)

5. Page 150, Exercise 5.

Technique (v)

6. Calculate $\mathbf{x}_v[f]$ where $f(x, y, z) = z$ and $\mathbf{x}$ is the geographical patch in the sphere.

Solutions

1. Page 149, Exercise 1.

The geographical patch is

$$\mathbf{x}(u, v) = (r \cos v \cos u, r \cos v \sin u, r \sin v).$$

(a) If $f(p) = p_1^2 + p_2^2$ then the coordinate expression is given by the composite

$$(u, v) \xmapsto{\mathbf{x}} \mathbf{x}(u, v) \xmapsto{f} r^2 \cos^2 v \cos^2 u + r^2 \cos^2 v \sin^2 u = r^2 \cos^2 v.$$

(b) If $f(p) = (p_1 - p_2)^2 + p_3^2$ then the coordinate expression is

$$\begin{aligned}(u, v) \longmapsto \mathbf{x}(u, v) \longmapsto & r^2 \cos^2 v(\cos u - \sin u)^2 + r^2 \sin^2 v \\ &= r^2 \cos^2 v(1 - 2\cos u \sin u) + r^2 \sin^2 v \\ &= r^2 (1 - 2\cos^2 v \cos u \sin u) \\ &= r^2 (1 - \cos^2 v \sin 2u).\end{aligned}$$

2. Page 149, Exercise 2.

The parametrization of the torus is

$$\mathbf{x}(u, v) = ((R + r \cos u)\cos v, (R + r \cos u)\sin v, r \sin u).$$

(a) We are given the coordinate functions for the curve α. They are $a_1(t) = a_2(t) = t$.

Hence

$$\begin{aligned}\alpha(t) &= \mathbf{x}(a_1(t), a_2(t)) = \mathbf{x}(t, t) \\ &= ((R + r \cos t) \cos t, (R + r \cos t) \sin t, r \sin t).\end{aligned}$$

(b) If $\alpha(t) = \alpha(t')$ then $\sin t = \sin t'$. If as well $\sin t = \sin t' \neq 0$ then $R + r\cos t = R + r\cos t'$ and hence $\cos t = \cos t'$. If on the contrary $\sin t = \sin t' = 0$ then $\cos t$, $\cos t' = \pm 1$. Since $(R + r\cos t)\cos t = (R + r\cos t')\cos t'$ and $r < R$ it follows that either $\cos t = 1 = \cos t'$ or $\cos t = -1 = \cos t'$. In any case both $\sin t = \sin t'$ and $\cos t = \cos t'$ and so $t - t' = 2\pi n$ for some integer n.

Conversely, if $t - t' = 2\pi n$ then $\alpha(t) = \alpha(t')$ since the trigonometric functions are periodic with period 2π.

Hence $\alpha(t) = \alpha(t')$ if and only if $t - t' = 2\pi n$ for some integer n; that is, α is periodic with period 2π.

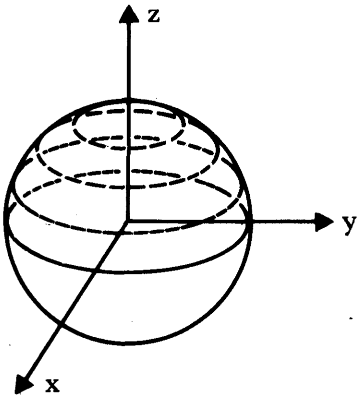
3. Page 150, Exercise 6.

(a)
$$x(u, v) = (u, v, (1-u^2-v^2)^{\frac{1}{2}})$$

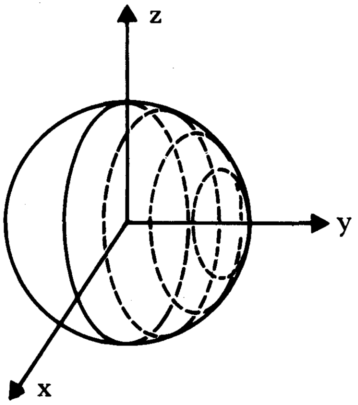
and

$$y(u, v) = (v, (1-u^2-v^2)^{\frac{1}{2}}, u).$$

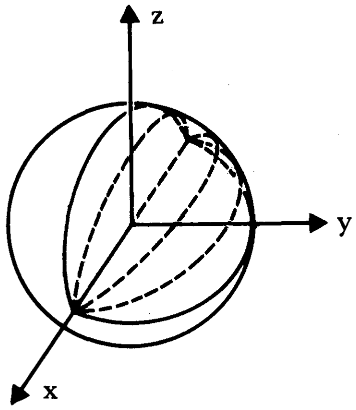
The first patch covers the northern hemisphere of the sphere, that is the hemisphere of points with strictly positive z coordinate. Similarly the second patch covers the hemisphere of points with strictly positive y coordinate. The intersection of these hemispheres is the region on the sphere for which both the z and y coordinates are strictly positive. These regions are shown in the following diagrams.



Region 1



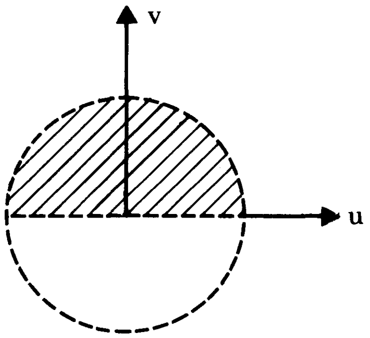
Region 2



Region 3

(b) $y^{-1}x$ is defined on points (u, v) belonging to D such that x(u, v) belongs to region 2 or equivalently region 3. For this x(u, v) must have positive y coordinate.

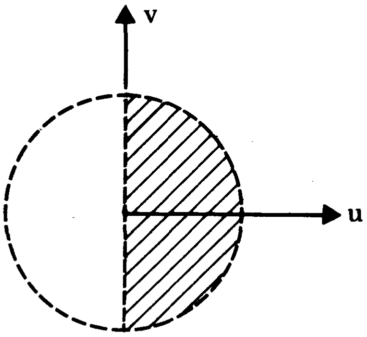
This happens when $v > 0$.



On this domain

$$y^{-1}x : (u, v) \stackrel{x}{\longmapsto} (u, v, (1-u^2-v^2)^{\frac{1}{2}}) \stackrel{y^{-1}}{\longmapsto} ((1-u^2-v^2)^{\frac{1}{2}}, u).$$

(c) A similar argument to that in (b) shows that $x^{-1}y$ is defined on points in D for which $y(u, v)$ has positive z coordinate. This happens if $u > 0$.



On this domain

$$x^{-1}y \; : \; (u, v) \longmapsto (v, (1-u^2-v^2)^{\frac{1}{2}}).$$

4. Page 150, Exercise 8(a).

By Result (vi), if $\alpha = x(a_1, a_2)$ is a curve in a surface then

$$\alpha' = x_u(a_1, a_2)a_1' + x_v(a_1, a_2)a_2'.$$

In this exercise $a_1(t) = \sqrt{2}t$, $a_2(t) = e^t$ and hence

$$\alpha'(t) = \sqrt{2}\, x_u(\sqrt{2}t, e^t) + e^t\, x_v(\sqrt{2}t, e^t).$$

5. Page 150, Exercise 5.

(a) M can be defined implicitly as

$$M : g(x, y, z) = z - f(x, y) = 0.$$

Hence by Lemma IV.3.8 we know that

$$\nabla g = \frac{\partial g}{\partial x} U_1 + \frac{\partial g}{\partial y} U_2 + \frac{\partial g}{\partial z} U_3 = -\frac{\partial f}{\partial x} U_1 - \frac{\partial f}{\partial y} U_2 + U_3$$

is a normal vector field on M.

Hence a tangent vector **v** at **p** is a tangent vector to M at **p** if and only if

$$\begin{aligned} 0 &= \nabla g(\mathbf{p}) \cdot \mathbf{v} \\ &= \left(-\frac{\partial f}{\partial x}(p_1, p_2), -\frac{\partial f}{\partial y}(p_1, p_2), 1\right) \cdot (v_1, v_2, v_3) \\ &= -v_1 \frac{\partial f}{\partial x}(p_1, p_2) - v_2 \frac{\partial f}{\partial y}(p_1, p_2) + v_3, \end{aligned}$$

which is the required condition.

(b) Since the vector $x_u(u_0, v_0) \times x_v(u_0, v_0)$ is nonzero and normal to M at $\mathbf{p} = x(u_0, v_0)$, the vector v at p is tangent to M if and only if

$$\mathbf{v} \cdot (x_u(u_0, v_0) \times x_v(u_0, v_0)) = 0.$$

6. By Result (ix) we know that

$$\mathbf{x}_v[f] = \frac{\partial}{\partial v}(f\mathbf{x}).$$

Now

$$(f\mathbf{x})(u, v) = r \sin v$$

and hence

$$\mathbf{x}_v[f] = r \cos v.$$

That is,

$$\mathbf{x}_v(u_0, v_0)[f] = r \cos v_0.$$

IV.4 DIFFERENTIAL FORMS ON A SURFACE

Introduction

This section follows on from Section IV.3 and is closely related to Section I.6.

In Section I.6 we described differential forms on E^3 in terms of coordinates. In this section we shall describe differential forms on a surface in a coordinate independent way. The results that we shall obtain parallel those of Section I.6.

READ: Section IV.4 (pages 152–157) omitting the proof of Lemma 4.5.

Comments

(i) *Differential Forms* A 2-form η operates only on pairs of tangent vectors with the same point of application. If v_p, w_p are vectors tangent to M at p then the real number $\eta(v_p, w_p)$ is defined.

Condition (1), of Definition 4.1, says that η restricted to any one tangent space $T_p(M)$ is a bilinear form, and condition (2) says that it is anti-symmetric. If **u**, **v**, **w** are tangent vectors to M at p and a, b, are real numbers then condition (1) is equivalent to

$$\eta(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = a\eta(\mathbf{u}, \mathbf{w}) + b\eta(\mathbf{v}, \mathbf{w})$$

and

$$\eta(\mathbf{u}, a\mathbf{v} + b\mathbf{w}) = a\eta(\mathbf{u}, \mathbf{v}) + b\eta(\mathbf{u}, \mathbf{w}).$$

As well as using the pointwise principle to define addition of forms we can use it to define the multiplication of a form by a real valued function f on M. If ϕ is a 1-form and η a 2-form on M and **v**, **w** are tangent vectors to M at p then we can define forms $f\phi$ and $f\eta$ as follows:

$$(f\phi)(\mathbf{v}) = f(p)\,\phi(\mathbf{v})$$

and

$$(f\eta)(\mathbf{v}, \mathbf{w}) = f(p)\,\eta(\mathbf{v}, \mathbf{w}).$$

We can also use the pointwise principle to define the effect of forms on tangent vector fields. If V, W are tangent vector fields on M and ϕ is a 1-form and η a 2-form then the real valued functions, $\phi(V)$ and $\eta(V, W)$, on M are defined as follows:

$$(\phi(V))(p) = \phi(V(p))$$

and

$$(\eta(V, W))(p) = \eta(V(p), W(p)),$$

for each point p belonging to M.

Since $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$ are a basis for the tangent vectors at the point $p = x(u_0, v_0)$, the 1-form ϕ and 2-form η are determined at p by the values of $\phi(x_u(u_0, v_0))$, $\phi(x_v(u_0, v_0))$ and $\eta(x_u(u_0, v_0), x_v(u_0, v_0))$ respectively.

We use the notation x_u and x_v to stand for the vector fields, on the image of the patch x, that take the values of $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$ at the point $p = x(u_0, v_0)$. Hence on the image of the patch x the forms ϕ and η are determined by the real valued functions $\phi(x_u)$, $\phi(x_v)$ and $\eta(x_u, x_v)$.

We have defined 0-, 1-, and 2-forms on a surface but we do not yet know if they ever exist. In fact it is easy to construct examples. Since 0-forms are just differentiable functions on the surface they certainly exist. We can always at least choose constant functions.

Suppose we are dealing with a simple surface $M = x(D)$. We can define a 1-form ϕ as follows:

$$\phi : v \longmapsto v \cdot x_u(u_0, v_0),$$

where v is a vector tangent to M at $p = x(u_0, v_0)$.

Similarly we can define a 2-form η as follows:

$$\eta : v, w \longmapsto (v \times w) \cdot (x_u(u_0, v_0) \times x_v(u_0, v_0)),$$

where v, w are vectors tangent to M at $p = x(u_0, v_0)$. It is not difficult to see that ϕ and η, defined above, are non-trivial forms on $M = x(D)$.

(ii) *The wedge product* When we form the wedge product of two differentials we omit the wedge symbol. For instance we write $du_1 du_2$ rather than $du_1 \wedge du_2$.

Supplementary Comments

(i) *Page 153: after Definition 4.3* We need to check that $\phi \wedge \psi$ defined in this way really is a 2-form (an antisymmetric bilinear form on each tangent space). This is straightforward. For instance condition (2) is verified as follows. For any tangent vectors v, w at p,

$$\begin{aligned}(\phi \wedge \psi)(v, w) &= \phi(v)\,\psi(w) - \phi(w)\,\psi(v), && \text{by Definition 4.3,}\\ &= -\,(\phi(w)\,\psi(v) - \phi(v)\,\psi(w))\\ &= -\,((\phi \wedge \psi)(w, v)), && \text{by Definition 4.3.}\end{aligned}$$

(ii) *Page 154: line 2* For any pair of tangent vectors v, w at p,

$$\begin{aligned}(\phi \wedge \psi)\,(v, w) &= \phi(v)\,\psi(w) - \phi(w)\,\psi(v), && \text{by Definition 4.3,}\\ &= -\,(\psi(v)\,\phi(w) - \psi(w)\,\phi(v))\\ &= -\,((\psi \wedge \phi)(v, w)), && \text{by Definition 4.3,}\\ &= (-(\psi \wedge \phi))(v, w).\end{aligned}$$

Since this is true for all pairs v, w it follows that

$$\phi \wedge \psi = -\,\psi \wedge \phi.$$

Summary

Definitions

(i)	0-form on M : $f, g \ldots$	Page 152, line 14 of the section
(ii)	1-form on M : $\phi, \psi \ldots$	Page 152, line 15 of the section
(iii)	2-form on M : $\eta, \xi \ldots$	Page 152, Definition 4.1
(iv)	Pointwise addition of forms on M	Page 152, line −2
(v)	Pointwise multiplication of forms and functions on M	Text, page 43
(vi)	Pointwise effect of forms on tangent vector fields on M	Text, page 43
(vii)	The wedge product of 1-forms on M	Page 153, Definition 4.3
(viii)	The exterior derivative of a 0-form (function) on M	Page 154, line 4
(ix)	The exterior derivative of a 1-form on M	Page 154, Definition 4.4

Results

(i)	For any 2-form η on M and any vector v tangent to M then $\eta(v, v) = 0$.	Page 153, line 6
(ii)	If v, w are (linearly independent) vectors tangent to M at p and η is a 2-form on M then $\eta(av + bw, cv + dw) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \eta(v, w)$.	Page 153, Lemma 4.2
(iii)	If ϕ, ψ are 1-forms on M then $\phi \wedge \psi = -\psi \wedge \phi$.	Page 154, line 2
(iv)	The definition of the exterior derivative of a 1-form on M is independent of the choice of patches.	Page 154, Lemma 4.5
(v)	The exterior derivative of a wedge product satisfies the linear and Leibnizian properties.	Page 155, line 20
(vi)	If f is a differentiable function on M then $d(df) = 0$.	Page 155, Lemma 4.6
(vii)	1-forms that agree on $\mathbf{x}_u$ and $\mathbf{x}_v$ are equal on the patch $\mathbf{x}(D)$.	Page 156, line 10
(viii)	2-forms that agree on the pair $(\mathbf{x}_u, \mathbf{x}_v)$ are equal on the patch $\mathbf{x}(D)$.	Page 156, line 12

Technique

(i) The evaluation of linear combinations, wedge products and exterior derivatives of forms on a surface, using the above definitions and results.

Exercise

1. Page 156, Example 4.7

There are three ways we can extend the exterior calculus to deal with E^2.

(a) We can treat it formally as in Section I.6.

(b) We can describe 1- and 2-forms as operating on tangent vectors and pairs of tangent vectors in E^2. The wedge product is defined by analogy with Definition IV.4.3 and the exterior derivative by analogy with Definition IV.4.4 (replacing $\mathbf{x}_u$ and $\mathbf{x}_v$ by the natural frame field U_1 and U_2).

(c) We can consider E^2 as a surface in E^3 given by the single patch $\mathbf{x} : E^2 \longrightarrow E^3$

$$(u, v) \longmapsto (u, v, 0).$$

Check the results of Example IV.4.7 in each of these three cases. (In case (c) let u_1, u_2 be the restrictions of the coordinate functions x, y to $\mathbf{x}(E^2)$.)

Solution

1. (a)

(1) A 1-form is an expression $f_1 du_1 + f_2 du_2$, where f_1 and f_2 are (differentiable) functions.

(2) A 2-form is an expression gdu_1du_2.

(3) The wedge product is defined using linearity and the alternation rule so

$$\begin{aligned}\phi \wedge \psi &= (f_1du_1 + f_2du_2) \wedge (g_1du_1 + g_2du_2)\\ &= f_1g_1du_1du_1 + f_1g_2du_1du_2 + f_2g_1du_2du_1 + f_2g_2du_2du_2\\ &= (f_1g_2 - f_2g_1)\,du_1du_2.\end{aligned}$$

(4) By analogy with Corollary I.5.5 we *define*

$$df = \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2.$$

(5) We define the exterior derivative of a 1- form by finding the exterior derivative of its coordinate functions.

That is

$$\begin{aligned}d\phi &= df_1 \wedge du_1 + df_2 \wedge du_2\\ &= \left(\frac{\partial f_1}{\partial u_1}du_1 + \frac{\partial f_1}{\partial u_2}du_2\right) \wedge du_1 + \left(\frac{\partial f_2}{\partial u_1}du_1 + \frac{\partial f_2}{\partial u_2}du_2\right) \wedge du_2\\ &= \left(\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2}\right)du_1du_2.\end{aligned}$$

(b)

(1) A 1-form is a mapping on tangent vectors (and hence vector fields) that is linear on each tangent vector space. Hence two 1-forms ϕ and ψ are equal if they agree on the natural frame fields U_1 and U_2.

By analogy with Definition I.5.2 we define the differential in terms of the directional derivative and hence

$$du_i(U_j) = U_j[u_i] = \delta_{ij} \quad \text{for } i, j \in \{1, 2\}.$$

Let $\psi = \phi(U_1)du_1 + \phi(U_2)du_2$. Then

$$\psi(U_1) = \phi(U_1)du_1(U_1) + \phi(U_2)du_2(U_1) = \phi(U_1)$$

and

$$\psi(U_2) = \phi(U_1)du_1(U_2) + \phi(U_2)du_2(U_2) = \phi(U_2)$$

and so

$$\phi = \psi = \phi(U_1)du_1 + \phi(U_2)du_2.$$

(2) Two 2-forms are equal if they agree on the pair of vector fields (U_1, U_2), by analogy with Lemma IV.4.2.

By analogy with Definition IV.4.3 we have

$$du_1du_2(U_1, U_2) = du_1(U_1)du_2(U_2) - du_1(U_2)du_2(U_1) = 1.$$

Let $\xi = \eta(U_1, U_2)du_1du_2$. Then

$$\xi(U_1, U_2) = \eta(U_1, U_2)du_1du_2(U_1, U_2) = \eta(U_1, U_2)$$

and so

$$\eta = \xi = \eta(U_1, U_2)\,du_1du_2.$$

(3) Since the wedge product satisfies the alternation rule this follows as in part (a) (3).

(4) By analogy with Definition I.5.2,

$$df(U_i) = U_i[f] \quad \text{for } i = 1, 2.$$

But the directional derivative $U_i\,[f]$ is just $\partial f/\partial u_i$.

By result (b) (1),

$$\begin{aligned} df &= df(U_1)du_1 + df(U_2)du_2 \\ &= \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2. \end{aligned}$$

(5) By analogy with Definition IV.4.4 we define

$$\begin{aligned} d\phi(U_1, U_2) &= \frac{\partial}{\partial u_1}(\phi(U_2)) - \frac{\partial}{\partial u_2}(\phi(U_1)) \\ &= \frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2}, \qquad \text{by result(b) (1).} \end{aligned}$$

Hence

$$d\phi = \left(\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2}\right)du_1\,du_2, \qquad \text{by result (b) (2).}$$

(c) We consider E^2 as the surface in E^3 given by the single patch

$$\begin{aligned} \mathbf{x} : E^2 &\longrightarrow E^3 \\ (u, v) &\longmapsto (u, v, 0). \end{aligned}$$

Then $\mathbf{x}_u = U_1$, restricted to $\mathbf{x}(E^2)$, and $\mathbf{x}_v = U_2$, again restricted to $\mathbf{x}(E^2)$. We can also consider the Euclidean coordinate functions x and y restricted to $\mathbf{x}(E^2)$. We call these u_1 and u_2. By analogy with Definition I.5.2 and the definition of the directional derivative, Definition IV.3.10, we find from first principles that

$$du_i(U_j) = U_j[u_i] = \delta_{ij} \quad \text{for } i, j \in \{1,2\}.$$

(1) Since two 1-forms on a surface are equal if they agree on $\mathbf{x}_u$ and $\mathbf{x}_v$ we find, on applying both sides of equation (1) to U_1 and U_2, that

$$\phi = \phi(U_1)du_1 + \phi(U_2)du_2,$$

for any 1-form ϕ on $\mathbf{x}(D)$.

(2) Using the definition of the wedge product, Definition IV.4.3, and the result that two 2-forms are equal if they agree on the pair (U_1, U_2) we find as in (b) (2) that

$$\eta = \eta(U_1, U_2)du_1\,du_2$$

(3) This follows by the alternation property.

(4) We can evaluate the differential df on each of the partial velocities $U_1(u_0, v_0, 0)$ and $U_2(u_0, v_0, 0)$ at $(u_0, v_0, 0) = \mathbf{x}(u_0, v_0)$. We find that

$$\begin{aligned} df(U_1(u_0, v_0, 0)) &= U_1(u_0, v_0, 0)[f] \\ &= \frac{d}{dt}(f(u_0 + t, v_0, 0))\Big|_{t=0}. \end{aligned}$$

We shall write this as $\dfrac{\partial f}{\partial u_1}(u_0, v_0, 0)$ and $df\,(U_2\,(u_0, v_0, 0))$ as $\dfrac{\partial f}{\partial u_2}(u_0, v_0, 0)$.

From (c) (1)

$$df = df(U_1)du_1 + df(U_2)du_2$$

and hence

$$df = \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2.$$

(5) Combining the arguments of parts (b) (5) and (c) (4) we find that for any 1-form ϕ on $x(E^2)$

$$d\phi = \left(\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2}\right) du_1\, du_2,$$

where $f_1 = \phi(U_1)$ and $f_2 = \phi(U_2)$.

IV.5 MAPPINGS OF SURFACES

Introduction

This section follows on from Section IV.4.

Having defined surfaces and their differential calculus we next look at mappings between these structures. The mappings we choose are those that compose with the patches to give differentiable mappings. Given such a mapping F from M to N it is possible to take a tangent vector to M and map it to a tangent vector to N. By contrast a differential form on N can be mapped to a differential form on M. This allows us to relate the differential calculus on one surface with that on another in such a way that the various structures that we have defined are preserved. That is, the wedge product of two forms is mapped to the wedge product of the images of the forms, and the exterior derivative of a form is mapped to the exterior derivative of the image of the form.

We introduce certain mappings that became of great interest during the 18th and 19th centuries when mathematicians were interested in the problem of trying to make plane maps of our spherical globe.

READ: Section IV.5 (pages 158 - 164).

(i) *The derivative map* As usual we need to check that F_* is well defined. That is, given any tangent vector v to M and any curve α in M with initial velocity v we need to check that the initial velocity of the curve $F(\alpha)$ in N depends only on v and not on the choice of α.

We prove this in the same way that we proved that the directional derivative (Definition IV.3.10) was well defined. Suppose we extend F to a Euclidean differentiable function G defined on some neighbourhood in E^3 of the point in question. If α is a curve in M and $\beta = F(\alpha) = G(\alpha)$ then, by Theorem I.7.8, $\beta' = G_*(\alpha')$. So the initial velocity $\beta'(0) = G_*(\alpha'(0)) = G_*(v)$, where v is the initial velocity of α. Now $G_*(v)$ depends only on G and v and not on the choice of α while $\beta = F(\alpha)$ depends only on F and α and not on the extension G. Hence both sides of the equation are independent of the choice of curve α or extension G. Hence if we define $F_*(v) = \beta'(0) = (F\alpha)'(0)$ then it is well defined.

Given any vector v tangent to M then by definition we can find a curve α of which it is the initial velocity and so we do have a mapping F_* taking $T_p(M)$ to $T_{F(p)}(N)$. Since we have seen that F_* can be defined using the derivative map G_* of some extension and we saw in Section I.7 that such derivatives were linear it follows that the restriction F_* must also be a linear transformation. It also follows that if F_1 is a mapping from M_1 to M_2 and F_2 is a mapping from M_2 to M_3 then $(F_2 \circ F_1)_* = F_{2*} \circ F_{1*}$.

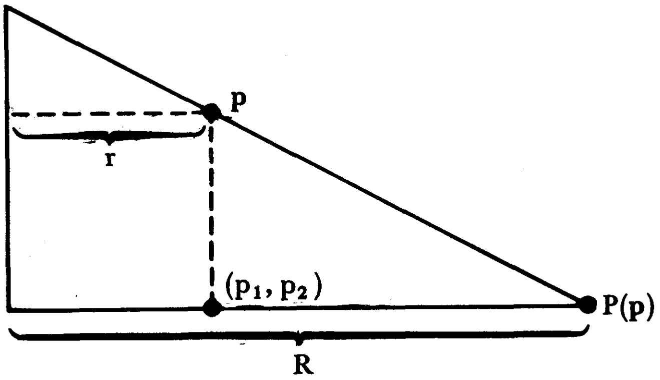
(ii) *F_* and F^** The positions of the stars on the symbols F_* and F^* are conventional. F is a map from M to N. Since tangent vectors are mapped in the same direction as F, from $T_p(M)$ to $T_{F(p)}(N)$, we use a lower star, F_*. Since differential forms are mapped in the opposite direction to F, that is forms on N are taken to forms on M, we use an upper star, F^*.

Since F_* is a linear transformation it is compatible with the vector space structures on the corresponding tangent spaces. Theorem 5.7 tells us that F^* is compatible with the operations of addition, wedge product and exterior derivative on the corresponding spaces of forms.

Supplementary Comments

(i) *Page 159: Example 5.2(1), line -8* The u-parameter measures the same angle for both the sphere and the cylinder. For the sphere the v-parameter measures the angle of elevation while for the cylinder it measures the height. Since F maps points radially onwards it preserves their height. A point on the sphere with v-parameter v_0 has height $\sin v_0$ and so is mapped by F to a point on the cylinder with v-parameter $\sin v_0$. The formula follows.

(ii) *Page 160: line 8*



By similar triangles $\frac{1}{R}P(p) = \frac{1}{r}(p_1, p_2)$.

Summary

Notation

F_*	Page 160, Definition 5.3
$F_{*p} : T_p(M) \longrightarrow T_{F(p)}(N)$	Page 161, line 12
F^*	Page 163, Definition 5.6

Definitions

(i)	Differentiable mapping between surfaces	Page 159, Definition 5.1
(ii)	Derivative map F_*	Page 160, Definition 5.3
(iii)	Regular	Page 161, line 11
(iv)	Diffeomorphism	Page 161, line -8
(v)	Induced mapping of forms F^*	Page 163, Definition 5.6

Examples

(i)	Horizontal projection of the sphere, minus poles, into the cylinder	Page 159, Example 5.2(1)
(ii)	Stereographic projection of the sphere, minus the north pole, onto the plane	Page 160, Example 5.2(2)

Results

(i)	F_* restricts to a linear transformation from $T_p(M)$ to $T_{F(p)}(N)$.	Page 160, line - 7
(ii)	The derivative map of a composite is the composite of the derivative maps; $(G \circ F)_* = G_* \circ F_*$.	Page 160, line - 2
(iii)	If x is a parametrization and $y = F(x)$ then $F_*(x_u) = y_u$ and $F_*(x_v) = y_v$.	Page 161, line 7
(iv)	If F is a one-to-one, onto and regular mapping then it is a diffeomorphism.	Page 162, line 1
(v)	For any mapping F the mapping F^*, between forms, is compatible with the operations of addition, wedge product and exterior derivative.	Page 163, Theorem 5.7

Techniques

(i)	Checking that a function is a mapping between surfaces by examining coordinate expressions.	Page 159, Definition 5.1
(ii)	Description of the derivative map in terms of partial velocities.	Page 161, line 7
(iii)	Showing that F is a diffeomorphism by checking that it is one-to-one, onto and regular.	Page 162, line 1
(iv)	Determination of the effect of F^* on forms.	Page 163, Definition 5.6, and Page 163, Theorem 5.7

Exercises

Technique (i)

1. Describe the mapping of Exercise 2(a), page 164, in terms of the geographical parametrization and hence describe its effect on meridians and parallels. (Restrict the domain of the parametrization to the region $-\frac{\pi}{2} < v < \frac{\pi}{2}$.)

Technique (ii)

2. Describe the derivative of the mapping in Exercise 1 in terms of the partial velocities associated with the usual geographical parametrization.

Technique (iii)

3. Check that the mapping of surfaces $P : \Sigma \longrightarrow E^2$ given in Example 5.5, page 162, is a diffeomorphism. (Use a Monge patch to prove that P is regular at the south pole.)

Technique (iv)

4. Page 165, Exercise 5. Find $A^*\gamma$, where γ is given by

$$\gamma(v_p, w_p) = p_3(v_1 w_2 - v_2 w_1).$$

5. Complete the proof of Theorem IV.5.7

Solutions

1. Page 164, Exercise 2(a).

$$F(p) = -p.$$

The geographical parametrization is

$$x(u, v) = r(\cos v \cos u, \cos v \sin u, \sin v).$$

Hence $F(x(u, v)) = r(-\cos v \cos u, -\cos v \sin u, -\sin v)$.

We want to describe the righthand side of this equation as $x(u', v')$. Now $\sin(-v) = -\sin v$, $\cos(-v) = \cos v$ and $\cos(u + \pi) = -\cos u$, $\sin(u + \pi) = -\sin u$. Hence $F(x(u, v)) = x(u + \pi, -v)$.

Hence the parallel given by the u-parameter curve $v = v_0$ is mapped to the parallel given by $v = -v_0$. A parallel in the northern hemisphere is mapped to the corresponding parallel in the southern hemisphere and vice versa.

The meridian given by the v-parameter curve $u = u_0$ is mapped to the v-parameter curve $u = u_0 + \pi$. That is, the meridians are rotated by 180°.

2. We must find the images of the two partial velocities at some point p and express these images in terms of the partial velocities at $F(p) = -p$. The geographical parametrization is

$$x(u, v) = r(\cos v \cos u, \cos v \sin u, \sin v).$$

The composite mapping $y = Fx = -x$ and hence

$$y(u, v) = -r(\cos v \cos u, \cos v \sin u, \sin v) = x(u + \pi, -v).$$

Suppose p is the point $x(u_0, v_0)$, then according to the result on page 161, line 7, the derivative $F_* : T_p(\Sigma) \longrightarrow T_{F(p)}(\Sigma)$ is given by

$$F_*(x_u(u_0, v_0)) = y_u(u_0, v_0)$$

and

$$F_*(x_v(u_0, v_0)) = y_v(u_0, v_0).$$

We want to express $y_u(u_0, v_0)$ and $y_v(u_0, v_0)$ in terms of the partial velocities of the geographical parametrization at their point of action $y(u_0, v_0) = x(u_0 + \pi, -v_0)$. That is, we want $y_u(u_0, v_0)$, $y_v(u_0, v_0)$ in terms of $x_u(u_0 + \pi, -v_0)$ and $x_v(u_0 + \pi, -v_0)$.

Now

$$\begin{aligned} y_u(u_0, v_0) &= (r \cos v_0 \sin u_0, -r \cos v_0 \cos u_0, 0) \\ &= (-r \cos(-v_0) \sin(u_0 + \pi), r \cos(-v_0) \cos(u_0 + \pi), 0) \\ &= x_u(u_0 + \pi, -v_0). \end{aligned}$$

Hence

$$F_*(x_u(u_0, v_0)) = x_u(u_0 + \pi, -v_0)$$

and a similar calculation shows that

$$F_*(x_v(u_0, v_0)) = -x_v(u_0 + \pi, -v_0).$$

Draw a picture for yourself: it may help you to understand the difference in sign of the two cases.

3. By Theorem IV.5.4 we need to check that P_* is one-to-one. For points other than the south pole this follows if we can show that y_u, y_v, where y is given on page 162, are linearly independent. Now

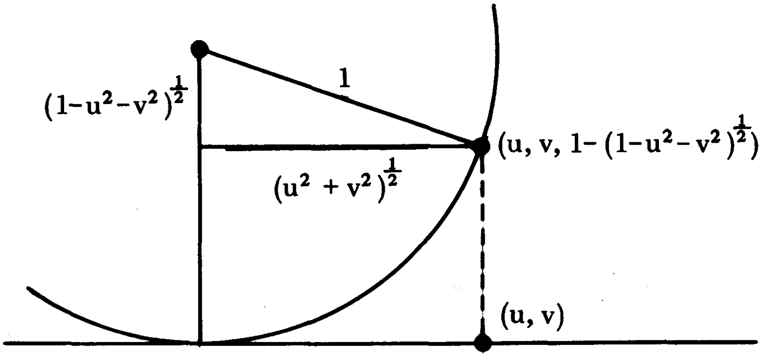
$$\begin{aligned} y_u &= \left(\frac{-2\cos v \sin u}{(1-\sin v)}, \frac{2\cos v \cos u}{(1-\sin v)} \right) \\ &= \frac{2\cos v}{(1-\sin v)} (-\sin u, \cos u), \end{aligned}$$

and

$$y_v = \left(\frac{2\cos u}{(1-\sin v)}, \frac{2\sin u}{(1-\sin v)} \right) = \frac{2}{(1-\sin v)} (\cos u, \sin u).$$

Since $v \in (-\frac{\pi}{2}, \frac{\pi}{2})$ it follows that $\cos v \neq 0$ and $\sin v \neq 1$. Hence y_u and y_v are well defined and nonzero. since the vectors $(\cos u, \sin u)$ and $(-\sin u, \cos u)$ are orthogonal y_u and y_v are linearly independent.

At the origin we use the following Monge patch:


$(1-u^2-v^2)^{\frac{1}{2}}$
1
$(u, v, 1-(1-u^2-v^2)^{\frac{1}{2}})$
$(u^2+v^2)^{\frac{1}{2}}$
(u, v)

$$x(u, v) = (u, v, 1 - (1 - u^2 - v^2)^{\frac{1}{2}}).$$

By the formula on page 160,

$$Px(u, v) = \left(\frac{2u}{1 + (1 - u^2 - v^2)^{\frac{1}{2}}}, \frac{2v}{1 + (1 - u^2 - v^2)^{\frac{1}{2}}} \right) = y(u, v).$$

Differentiation yields

$$y_u(0, 0) = (1, 0)$$

and

$$y_v(0, 0) = (0, 1),$$

which are linearly independent. P is regular at the south pole and hence $P : \Sigma_0 \longrightarrow E^2$ is a diffeomorphism.

4. Page 165, Exercise 5.

We have already proved, using the geographical patch, that

$$A_*(x_u(u_0, v_0)) = x_u(\pi + u_0, -v_0)$$

and

$$A_*(x_v(u_0, v_0)) = - x_v(\pi + u_0, -v_0).$$

If $v_p = x_u(u_0, v_0) = (-r \cos v_0 \sin u_0, r \cos v_0 \cos u_0, 0)$ then

$$x_u(\pi + u_0, -v_0) = (r \cos v_0 \sin u_0, -r \cos v_0 \cos u_0, 0) = (-v)_{-p}.$$

Similarly if $v_p = x_v(u_0, v_0)$ then $-x_v(\pi + u_0, -v_0) = (-v)_{-p}$. Since $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$ form a basis for the tangent vectors at $p = x(u_0, v_0)$ it follows that

$$A_*(v_p) = (-v)_{-p}.$$

This holds for each point p of the geographical patch. Since we can find similar patches to cover the remainder of Σ it holds for all points p in Σ. Hence A_* is one-to-one and A is a diffeomorphism.

We can prove that $A_*(v_p) = (-v)_{-p}$ more simply by extending the mapping A to all of E^3. Since $A(p_1, p_2, p_3) = (-p_1, -p_2, -p_3)$ it follows that A_* has Jacobian matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Hence $A_*(v_p) = (-v)_{-p}$, for all tangent vectors to E^3, and so restricting A to Σ we still have

$$A_*(v_p) = (-v)_{-p}.$$

Now if a 2-form is defined on Σ by

$$\gamma(v_p, w_p) = p_3(v_1 w_2 - v_2 w_1)$$

then $A^*\gamma$ is given by

$$\begin{aligned} A^*\gamma(v_p, w_p) &= \gamma(A_*(v_p), A_*(w_p)) \\ &= \gamma((-v)_{-p}, (-w)_{-p}) \\ &= (-p_3)((-v_1)(-w_2) - (-v_2)(-w_1)) \\ &= -p_3(v_1 w_2 - v_2 w_1) \\ &= -\gamma(v_p, w_p). \end{aligned}$$

Hence $A^*\gamma = -\gamma$.

5. (1) The proofs are routine. For instance if ξ and η are 1-forms then for any vector v tangent to M

$$\begin{aligned} (F^*(\xi + \eta))(v) &= (\xi + \eta)(F_*v), && \text{by the definition of } F^*, \\ &= \xi(F_*v) + \eta(F_*v), && \text{by the definition of the addition of forms,} \\ &= (F^*\xi)(v) + (F^*\eta)(v) \\ &= (F^*\xi + F^*\eta)(v), \end{aligned}$$

and hence $F^*(\xi + \eta) = F^*\xi + F^*\eta$.

(2) This result also follows automatically. For instance if ξ and η are 1-forms then for any pair of vectors v, w tangent to M at the same point

$$\begin{aligned} (F^*(\xi \wedge \eta))(v, w) &= (\xi \wedge \eta)(F_*v, F_*w) \\ &= \xi(F_*v)\eta(F_*w) - \xi(F_*w)\,\eta(F_*v) \\ &= F^*\xi(v)\,F^*\eta(w) - F^*\xi(w)F^*\eta(v) \\ &= (F^*\xi \wedge F^*\eta)(v, w) \end{aligned}$$

and hence $F^*(\xi \wedge \eta) = F^*\xi \wedge F^*\eta$.

If f is a 0-form and ϕ is a 1-form then for any tangent vector v at p

$$\begin{aligned}(F^*(f\phi))(v) &= (f\phi)(F_*(v))\\ &= f(F(p))\,\phi(F_*(v))\\ &= (F^*f)(p)(F^*\phi)(v)\\ &= ((F^*f)(F^*\phi))(v)\end{aligned}$$

and hence $F^*(f\phi) = (F^*f)\,(F^*\phi)$.

(3) Since we have already proved this result for 1-forms we need to prove it only for a 0-form (differentiable function) g. We need to prove $F^*(dg) = d(F^*(g))$.

For any vector v tangent to M we have

$$\begin{aligned}(F^*(dg))(v) &= dg(F_*v), && \text{by the definition of } F^*,\\ &= F_*v[g], && \text{by the definition of } dg,\\ &= \frac{d}{dt}(g(\beta(t)))\Big|_{t=0},\end{aligned}$$

where β is any curve in N with initial velocity F_*v. By the definition of F_*v we can choose β to be $F(\alpha)$ for some curve α in M with initial velocity v. So

$$\begin{aligned}(F^*(dg))(v) &= \frac{d}{dt}(g \circ F \circ \alpha(t))\Big|_{t=0}\\ &= v[gF], && \text{by the definition of the directional derivative,}\\ &= v[F^*(g)], && \text{by the definition of } F^* \text{ of a 0-form,}\\ &= d(F^*(g))(v), && \text{by the definition of the differential.}\end{aligned}$$

Since this is true for all tangent vectors v it follows that

$$F^*(dg) = d(F^*(g)).$$

IV.7 ORIENTABILITY

Introduction

For this section you need to recall the cross product from Section II.1, the gradient from Section IV.3 and differential forms from Section IV.4.

This last section is rather different from those that have preceded it. We have given a description of surfaces and objects associated with them in terms of local patches. In this section we look at a global property of a surface that can be determined only by examining all of the surface and not just by looking at a single patch. The property of *orientability*, one- or two-sidedness, is defined in terms of differential forms. This shows that these forms are not simply algebraic gadgets but do contain geometric information.

READ: Section IV.7 from page 177 line −6 to the end of page 178.

Comments

(i) *The normal vector field* In Definition 7.4 and Theorem 7.5 we are assuming that the nonvanishing 2-form and the nonvanishing normal vector field are both differentiable. We need to check that the vector field $Z(p) = v \times w/\mu(v, w)$ is differentiable. Assuming that this is defined independently of the choice of v and w we see that Z gives us a normal vector field

$$Z = \frac{x_u \times x_v}{\mu(x_u, x_v)}$$

on the image of a patch x. Since the patch x is differentiable this is a differentiable nonzero normal vector field on x(D). Now any point p in the surface has some neighbourhood that is the image of a patch and hence Z is differentiable on a neighbourhood of p. Hence Z is differentiable on all of the surface.

(ii) *Page 178: line −12, Lemma IV.3.8* Suppose the implicitly defined subset $M : g = c$ is a surface: then, since the converse of Lemma IV.3.8 is not true, it is not necessarily true that ∇g is a nonzero normal vector field and it may not be true that the surface is orientable. We always need to check that dg is nonzero to ensure these results.

Summary

Definition

(i)	Orientable	Page 177, Definition 7.4

Example

(i)	Möbius band	Page 180, Exercise 7

Results

(i)	A surface is orientable if and only if there exists a normal vector field that is nonzero at each point.	Page 178, Theorem 7.5
(ii)	The implicitly defined surface $M : g = c$ is orientable if ∇g is never zero on M.	Page 178, line -12
(iii)	There exist nonorientable surfaces.	Page 178, line -9

Technique

(i)	Determining a nowhere zero 2-form on an implicitly defined surface.	Page 178, Theorem 7.5, and Page 178, line -12

Exercises

Technique (i)

1. Find a nonvanishing 2–form on the sphere $M : x^2 + y^2 + z^2 = c^2$.

Theory Exercise

2. Fill in the details of the proof of Theorem IV.7.5.

Solutions

1. The surface is given by $M : x^2 + y^2 + z^2 = c^2$ and by Lemma IV.3.8 a nonvanishing normal vector field is given by

$$\nabla(x^2 + y^2 + z^2) = 2xU_1 + 2yU_2 + 2zU_3.$$

That is, we can take $Z(p) = 2p$.

Hence by Theorem IV.7.5 a nonvanishing 2-form is given by

$$\begin{aligned}\mu(v, w) &= Z(p)\cdot v \times w \\ &= 2p\cdot v \times w\end{aligned}$$

where v, w are tangent vectors to M at p.

2. (a) Suppose $\mu(v, w) = Z(p)\cdot v \times w$: then to prove that μ is a 2-form we need to check conditions (1) and (2) of Definition IV.4.1. These conditions follow from the analogous properties of the cross product.

(b) If Z(p) is defined by $Z(p) = v \times w/\mu(v, w)$ we need to show that it is independent of the choice of tangent vectors v and w. Suppose $\overline{v}, \overline{w}$ is another pair of linearly independent tangent vectors. Then since the tangent space is of dimension 2 there exist scalars a, b, c, d such that

$$\begin{aligned}\overline{v} &= av + bw \\ \overline{w} &= cv + dw.\end{aligned}$$

Then

$$\overline{Z}(\mathbf{p}) = \frac{\overline{\mathbf{v}} \times \overline{\mathbf{w}}}{\mu(\overline{\mathbf{v}}, \overline{\mathbf{w}})} = \frac{(a\mathbf{v} + b\mathbf{w}) \times (c\mathbf{v} + d\mathbf{w})}{\mu(a\mathbf{v} + b\mathbf{w}, c\mathbf{v} + d\mathbf{w})}.$$

Now $(a\mathbf{v} + b\mathbf{w}) \times (c\mathbf{v} + d\mathbf{w})$

$$= ac\mathbf{v} \times \mathbf{v} + ad\mathbf{v} \times \mathbf{w} + bc\mathbf{w} \times \mathbf{v} + bd\, \mathbf{w} \times \mathbf{w},$$

by the bilinearity of the cross product,

$$= ad\, \mathbf{v} \times \mathbf{w} + bc\, \mathbf{w} \times \mathbf{v}, \qquad \text{since } \mathbf{v} \times \mathbf{v} = \mathbf{w} \times \mathbf{w} = \mathbf{0},$$

$$= (ad - bc)(\mathbf{v} \times \mathbf{w}), \qquad \text{since } \mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}.$$

Now by Lemma IV.4.2

$$\mu(a\mathbf{v} + b\mathbf{w}, c\mathbf{v} + d\mathbf{w}) = (ad - bc)\, \mu(\mathbf{v}, \mathbf{w}).$$

Hence

$$\overline{Z}(\mathbf{p}) = \frac{(ad - bc)\, (\mathbf{v} \times \mathbf{w})}{(ad - bc)\, \mu(\mathbf{v}, \mathbf{w})}.$$

Since $\overline{\mathbf{v}}$, $\overline{\mathbf{w}}$ are linearly independent $ad - bc \neq 0$ and so the above expression is well defined and

$$\overline{Z}(\mathbf{p}) = \frac{\mathbf{v} \times \mathbf{w}}{\mu(\mathbf{v}, \mathbf{w})} = Z(\mathbf{p}).$$

The vector field is independent of the choice of **v** and **w**.

To see that it is a normal vector field let **u** be any tangent vector to M at **p**. Then there exist scalars a, b such that $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$, since the tangent space is 2-dimensional. So

$$\begin{aligned} \mathbf{u} \cdot Z(\mathbf{p}) &= \frac{(a\mathbf{v} + b\mathbf{w}) \cdot \mathbf{v} \times \mathbf{w}}{\mu(\mathbf{v}, \mathbf{w})} \\ &= \frac{a(\mathbf{v} \cdot \mathbf{v} \times \mathbf{w}) + b(\mathbf{w} \cdot \mathbf{v} \times \mathbf{w})}{\mu(\mathbf{v}, \mathbf{w})} \\ &= 0. \end{aligned}$$

Hence $Z(\mathbf{p})$ is perpendicular to each tangent vector to M at **p** and so Z is a normal vector field.

FURTHER EXERCISES AND SOLUTIONS

Section IV.1

Technique Exercises

Pages 131 - 133, Exercises 3, 7, 8, 9, 12.

Other Recommended Exercises

Page 131, Exercise 2.	This deals with descriptions of planes.
Page 132, Exercise 10. Page 133, Exercise 11.	These deal with the images of surfaces under simple mappings.

Solutions

3(a). Page 204, Fig. 5.18 inverted

3(b). Page 204, Fig. 5.19.

3(c). Page 205, Fig. 5.20 straightened out.

3(d). The xy plane.

7. Straightforward.

8. If $(u^2, uv, v^2) = (u'^2, u'v', v'^2)$, with $u', u, v', v > 0$, then $u = u'$ and $v = v'$, so x is one-to-one.

If (p_1, p_2, p_3) is in $x(D)$ then $x^{-1}(p_1, p_2, p_3) = (\sqrt{p_1}, \sqrt{p_3})$. Since the square root function, on the set of positive reals, is continuous it follows that x^{-1} is continuous.

Since the Jacobian matrix of x is

$$\begin{pmatrix} 2u & 0 \\ v & u \\ 0 & 2v \end{pmatrix}$$

which has rank 2 when $u, v > 0$ it follows that x is regular and hence a proper patch.

9. If $u + v = u' + v'$ and $u - v = u' - v'$ then $u = u'$ and $v = v'$. Hence x is one-to-one.

The Jacobian matrix of x is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ v & u \end{pmatrix}$$

which has rank 2 and hence x is regular.

The inverse mapping of $\mathbf{x}$ is

$$\mathbf{x}^{-1}\,(p_1, p_2, p_3) = \left(\frac{p_1 + p_2}{2}, \frac{p_1 - p_2}{2}\right),$$

which is continuous. Hence $\mathbf{x}$ is a proper patch.

Since $((u + v)^2 - (u - v)^2)/4 = uv$ the image of $\mathbf{x}$ belongs to M. Conversely if (p_1, p_2, p_3) belongs to M then it also belongs to the image of $\mathbf{x}$ since

$$(p_1, p_2, p_3) = (u_0 + v_0, u_0 - v_0, u_0 v_0),$$

where $u_0 = (p_1 + p_2)/2$ and $v_0 = (p_1 - p_2)/2$. Hence M is the image of $\mathbf{x}$.

12. When we revolve we preserve the x coordinate and the square of the distance from the x axis. Hence M is the surface $M : f(x, y^2 + z^2) = c = g(x, y, z)$. Using the chain rule and the fact that C is a curve we can show that dg is never zero on M and hence M is a surface.

2. $M : (a, b, c) \cdot \left((x, y, z) - \dfrac{d(a, b, c)}{a^2 + b^2 + c^2}\right) = 0.$

10. If $\mathbf{p}$ is a point of F(M) then $\mathbf{p} = F(\mathbf{q})$ for some point $\mathbf{q}$ in M. Since M is a surface there is a proper patch $\mathbf{x}$ giving a neighbourhood of $\mathbf{q}$ in M. We must show that $F(\mathbf{x})$ is a proper patch containing $\mathbf{p}$ in F(M).

Since $\mathbf{x}$ is one-to-one and F, an isometry, is also one-to-one it follows that $F(\mathbf{x})$ is one-to-one. Since $\mathbf{x}$ is regular, F is an isometry and $(F(\mathbf{x}))_* = F_* \mathbf{x}_*$ it follows that $(F(\mathbf{x}))_*$ is one-to-one and hence $F(\mathbf{x})$ is regular. Since $\mathbf{x}$ is a proper patch, F is an isometry and $(F(\mathbf{x}))^{-1} = \mathbf{x}^{-1}\, F^{-1}$ it follows that $(F(\mathbf{x}))^{-1}$ is continuous and hence $F(\mathbf{x})$ is a proper patch. So F(M) is a surface.

11.
$$\begin{aligned} d\bar{g}(F_* v) &= (F_* v)[\bar{g}] \\ &= v[\bar{g}(F)], \qquad \text{by Exercise I.7.9,} \\ &= v[g \circ F^{-1} \circ F] \\ &= v[g] \\ &= dg(v) \neq 0. \end{aligned}$$

Section IV.2

Technique Exercises

Pages 140-143, Exercises 3, 5, 8, 9, 10, 11.

Page 143, Exercise 13. This deals with simplification of parametrizations of surfaces of revolution.

Solutions

3. A profile curve is $C : (x - 4)^2 + z^2 = 4$, in the xz plane. The axis of revolution is the z axis.

5. $\mathbf{x}_u \times \mathbf{x}_v = v\delta' \times \delta$

This is nonzero, and hence $\mathbf{x}$ is regular, if and only if v and $\delta \times \delta'$ are nonzero.

8(a).

$$\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ \cos v & 1 & 0 \\ -u \sin v & 0 & \cos v \end{vmatrix}$$

$$= \cos v\, U_1 - \cos^2 v\, U_2 + u \sin v\, U_3$$

$$\neq 0 \text{ if } u > 0.$$

8(b). $M : y - (x^2 + y^2 z^2)^{\frac{1}{2}} = 0.$

8(c). $\mathbf{x}(u, v) = (0, 0, \sin v) + u(\cos v, 1, 0).$

9(a). The direction of L is $\beta'(u) + U_3$.

Hence $\mathbf{x}(u, v) = \beta(u) + v(\beta'(u) + U_3).$

9(b). $\beta(u) = (\cos u, \sin u, 0).$

$\mathbf{x}(u, v) = (\cos u - v \sin u, \sin u + v \cos u, v).$

9(c). The above points belong to $M' : x^2 + y^2 - z^2 = 1$. On M' we have $x^2 + y^2 = 1 + z^2$ and so we can write

$$x = (1 + v^2)^{\frac{1}{2}} \cos t$$

$$y = (1 + v^2)^{\frac{1}{2}} \sin t, \text{ where } z = v.$$

Let α be such that $\cos \alpha = (1 + v^2)^{-\frac{1}{2}}$, $\sin \alpha = v(1 + v^2)^{-\frac{1}{2}}$.

Let $t = u + \alpha$: then

$$x = \cos u - v \sin u$$

$$y = \sin u + v \cos u.$$

Hence M is the implicitly defined surface M'.

9(d). $\mathbf{x}(u, v) = (\cos u - v \sin u, \sin u + v \cos u, - v)$. $M : x^2 + y^2 - z^2 = 1.$

9(e). See Fig. 5.38, page 232.

10(a). $dg = \frac{2x}{a^2} dx + \frac{2y}{b^2} dy + \frac{2z}{c^2} dz \neq 0$ on M.

$\mathbf{x}_u \times \mathbf{x}_v = (-bc \cos^2 u \cos v, - ac \cos^2 u \sin v, - ab \sin u \cos u).$

So if $-\frac{\pi}{2} < u < \frac{\pi}{2}$ then $\cos u$ is never zero and $\mathbf{x}_u \times \mathbf{x}_v$ is never zero.

The image of $\mathbf{x}$ is the whole of M apart from the two points $(0, 0, \pm c)$.

10(b). $dg = \frac{2x}{a^2} dx + \frac{2y}{b^2} dy - \frac{2z}{c^2} dz \neq 0$ on M.

$\mathbf{x}_u \times \mathbf{x}_v = (-bc \cosh^2 u \cos v, - ac \cosh^2 u \sin v, ab \cosh u \sinh u)$, which is never zero, as $\cosh u$ is never zero.

The image of $\mathbf{x}$ is the whole of M.

10(c). $dg = \frac{2x}{a^2} dx + \frac{2y}{b^2} dy - \frac{2z}{c^2} dz \neq 0$ on M.

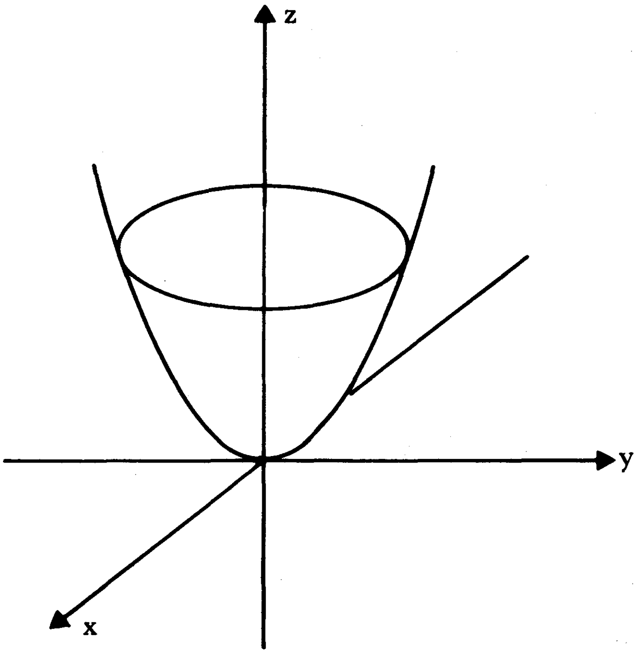
$\mathbf{x}_u \times \mathbf{x}_v = (-bc \sinh^2 u \cos v, - ac \sinh^2 u \sin v, ab \cosh u \sinh u)$, which is never zero, since $\sinh u$ is never zero $(u \neq 0)$.

The image of $\mathbf{x}$ is the whole of M apart from the two points $(0, 0, \pm c)$.

11(a). $dg = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy - dz \neq 0.$

$x_u \times x_v = (-2bu^2\cos v, -2au^2\sin v, abu)$, which is never zero, since $u > 0$. The parametrization omits only the point (0, 0, 0).

11(b). The u-parameter curves $v = v_0$ are parabolas in vertical planes through the z axis. The v-parameter curves $u = u_0$ are horizontal ellipses about the z axis.



13(a). If g' is never zero it has an inverse function k, defined on some suitable interval. The curve can be reparametrized as

$$u \longmapsto (g(k(u)), h(k(u)), 0) = (u, f(u), 0).$$

This gives the required parametrization of M.

13(b). As for (a).

Section IV.3

Technique Exercises

Pages 150 - 151, Exercises 7, 8(b), 9, 10.

Solutions

7. $M : xy - z = 0 = g(x, y, z).$

A normal vector field is $\nabla g = yU_1 + xU_2 - U_3$.

The surface is given by the patch $x(u, v) = (u, v, uv)$.

Linearly independent tangent vector fields are

$$x_u = (1, 0, v) = (1, 0, y)$$
$$x_v = (0, 1, u) = (0, 1, x).$$

8(b). $\alpha' = \sqrt{2}\, x_u + e^t x_v$

$$\alpha' \cdot x_u/\|x_u\| = \sqrt{2}\|x_u\| = \sqrt{2}\|e^t(-\sin\sqrt{2}t, \cos\sqrt{2}t, 0)\|$$

$$= \sqrt{2}e^t$$

$$\alpha' \cdot x_v/\|x_v\| = e^t \|x_v\| = e^t \| (\cos\sqrt{2}t, \sin\sqrt{2}t, 1) \|$$

$$= \sqrt{2}e^t.$$

9(a). The plane $\overline{T}_p(M)$ consists of all those points q such that

$$(q - p) \cdot z = 0.$$

If v is a tangent to M at p then p + v belongs to $\overline{T}_p(M)$ since

$$(p + v - p) \cdot z = v \cdot z = 0.$$

Conversely if q belongs to $\overline{T}_p(M)$ then since $(q - p) \cdot z = 0$ the vector $v = q - p$ is a tangent vector. Hence $q = p + v$ for some tangent vector v.

9(b). $x_u \times x_v$ is a never zero normal vector field.

9(c). ∇g is a never zero normal vector field if dg is never zero.

10(a). $(r - p) \cdot \nabla g(p) = 0$ gives the plane $z = 0$.

10(b). $((x, y, z) - (1, -2, 3)) \cdot (\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}) = \frac{x}{2} - \frac{y}{4} + \frac{z}{3} - 2 = 0.$

10(c). $((x, y, z) - (\sqrt{2}, \sqrt{2}, \frac{\pi}{2})) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \times (-\sqrt{2}, \sqrt{2}, 2) = 0$

$$((x, y, z) - (\sqrt{2}, \sqrt{2}, \tfrac{\pi}{2})) \cdot (\sqrt{2}, -\sqrt{2}, 2) = 0$$

$$\sqrt{2}x - \sqrt{2}y + 2z = \pi.$$

Section IV.4

Technique Exercises

Pages 157 - 158, Exercises 3, 5, 6, 9, 10.

Solutions

3.
$$\begin{aligned} d(fg)(v_p) &= v_p[fg] = v_p[f]\ g(p) + f(p)\, v_p[g] \\ &= df(v_p)g(p) + f(p)dg(v_p) \\ &= (df\, g + f\, dg)(v_p). \end{aligned}$$

$$\begin{aligned} d(f\phi)(x_u, x_v) &= \frac{\partial}{\partial u}(f\phi(x_v)) - \frac{\partial}{\partial v}(f\phi(x_u)) \\ &= \frac{\partial}{\partial u}(f(x)\phi(x_v)) - \frac{\partial}{\partial v}(f(x)\phi(x_u)) \\ &= f(x)\left(\frac{\partial}{\partial u}(\phi(x_v)) - \frac{\partial}{\partial v}(\phi(x_u))\right) \\ &\quad + \frac{\partial(f(x))}{\partial u}\phi(x_v) - \frac{\partial(f(x))}{\partial v}\phi(x_u) \\ &= f(x)\, d\phi(x_u, x_v) \\ &\quad + df(x_u)\phi(x_v) - df(x_v)\phi(x_u) \\ &= (df \wedge \phi + f d\phi)(x_u, x_v). \end{aligned}$$

5. By analogy with Lemma I.5.7

$$v_p[g(f)] = g'(f)(p)v_p[f]$$

and hence

$$d(g(f))(v_p) = (g'(f)df)(v_p).$$

6(a). $d(fgh) = df\,gh + f\,dgh$

$$= df\,gh + f\,dg\,h + fg\,dh.$$

6(b). $d(\phi f) = d(f\phi) = df \wedge \phi + fd\phi = fd\phi - \phi \wedge df.$

6(c). $(df \wedge dg)(v, w) = df(v)dg(w) - df(w)dg(v)$

$$= v[f]\,w[g] - v[g]\,w[f].$$

9(a). If $\pi_1 : E^2 \longrightarrow R$ is given by

$$\pi_1(u, v) = u$$

then $\tilde{u} = \pi_1 x^{-1}$.

This is differentiable on $x(D)$ if the coordinate expression

$$\tilde{u}y = \pi_1 x^{-1} y$$

is differentiable for all patches y. It is, because π_1 and $x^{-1}y$ are both differentiable.

Also $\tilde{u}(x(u, v)) = \pi_1 x^{-1} x(u, v) = \pi_1(u, v) = u.$

9(b). $d\tilde{u}(x_u) = x_u[\tilde{u}] = \frac{\partial}{\partial u}(\tilde{u}x) = \frac{\partial}{\partial u}(\pi_1) = 1$, etc.

9(c). $\phi(x_u) = (\phi(x_u)d\tilde{u} + \phi(x_v)d\tilde{v})(x_u)$, etc,

and

$$\eta(x_u, x_v) = (\eta(x_u, x_v)d\tilde{u}d\tilde{v})(x_u, x_v),$$

since

$$d\tilde{u}d\tilde{v}(x_u, x_v) = d\tilde{u}(x_u)d\tilde{v}(x_v) - d\tilde{u}(x_v)d\tilde{v}(x_u) = 1.$$

10(a). The coordinate system gives the radius and the angle the radius makes with the x axis.

10(b). The coordinate system is the natural coordinate system.

10(c). The coordinate system gives the longitude and latitude.

Section IV.5

Technique Exercises

Page 165, Exercises 6 and 7.

Other Recommended Exercises

Page 165, Exercise 4. Read Exercise 3 and use it to construct a mapping of the helicoid to the torus.

Page 166, Exercise 11. Read Exercise 10 and use it to construct mappings of the torus to itself.

Page 166, Exercise 14(c). This deals with the effect of composite mappings on forms.

Page 166, Exercise 16. This deals with stereographic projection.

Solutions

6(a). Let U_1, U_2 be the natural frame field on E^2.

Then

$$(x^*(\phi))(U_1) = \phi(x_*U_1) = \phi(x_u) \text{ and}$$

$$(x^*(\phi))(U_2) = \phi(x_v).$$

6(b). $(x^*(\upsilon))(U_1, U_2) = \upsilon(x_*U_1, x_*U_2) = \upsilon(x_u, x_v)$

6(c). $(x^*(d\phi))(U_1, U_2) = d\phi(x_u, x_v)$

$$= \left(\frac{\partial}{\partial u}(\phi x_v) - \frac{\partial}{\partial v}(\phi x_u)\right).$$

7(a). $x(u, v) = (r \cos v \cos u,\ r \cos v \sin u,\ r \sin v)$

$x_u = (-r \cos v \sin u,\ r \cos v \cos u,\ 0)$

$x_v = (-r \sin v \cos u,\ -r \sin v \sin u,\ r \cos v).$

Hence

$$\phi(x_u) = r^2 \cos^2 v,\ \phi(x_v) = 0$$

and

$$x^*(\phi) = r^2 \cos^2 v \, du$$

$$x^*(d\phi) = 2r^2 \cos v \sin v \ du\, dv.$$

7(b). $x^*(\upsilon) = r^3 \cos v \sin^2 v \ du\, dv.$

4. $(u \cos v,\ u \sin v,\ bv)$

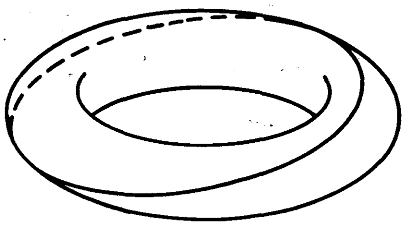
$\longmapsto ((R + r \cos u)\cos v,\ (R + r \cos u)\sin v,\ r \sin u).$

11(a). This maps the meridians three times round themselves. Not a diffeomorphism.

11(b). This rotates each meridian circle by 180°. A diffeomorphism.

11(c). This interchanges parallels and meridians. A diffeomorphism.

11(d). This maps meridians and parallels to curves that wrap round the torus as in the following diagram. Not a diffeomorphism.



14(c).
$$
\begin{aligned}
(GF)^*(f) &= f(GF) = F^*(fG) = F^*G^*(f).\\
((GF)^*\phi)(v) &= \phi((GF)_*v) = \phi(G_*(F_*(v)))\\
&= (G^*\phi)(F_*(v)) = ((F^*G^*)\phi)(v).\\
((GF)^*\eta)(v, w) &= \eta((GF)_*v, (GF)_*w) = \eta(G_*(F_*(v)), G_*(F_*(w)))\\
&= (G^*\eta)(F_*v, F_*w) = ((F^*G^*)\eta)(v, w).
\end{aligned}
$$

16(a). Straightforward.

16(b). P^{-1} gives a patch covering all of the sphere apart from one point. Another similar patch completes the covering of the sphere.

Section IV.7

Technique Exercises

Pages 180 - 181, Exercises 7, 8 (Is M^* orientable?).

Erratum: The last line on page 180 should read

"$x(u, v) = \beta(u) + v\delta(u), -\frac{1}{2} < v < \frac{1}{2}$,".

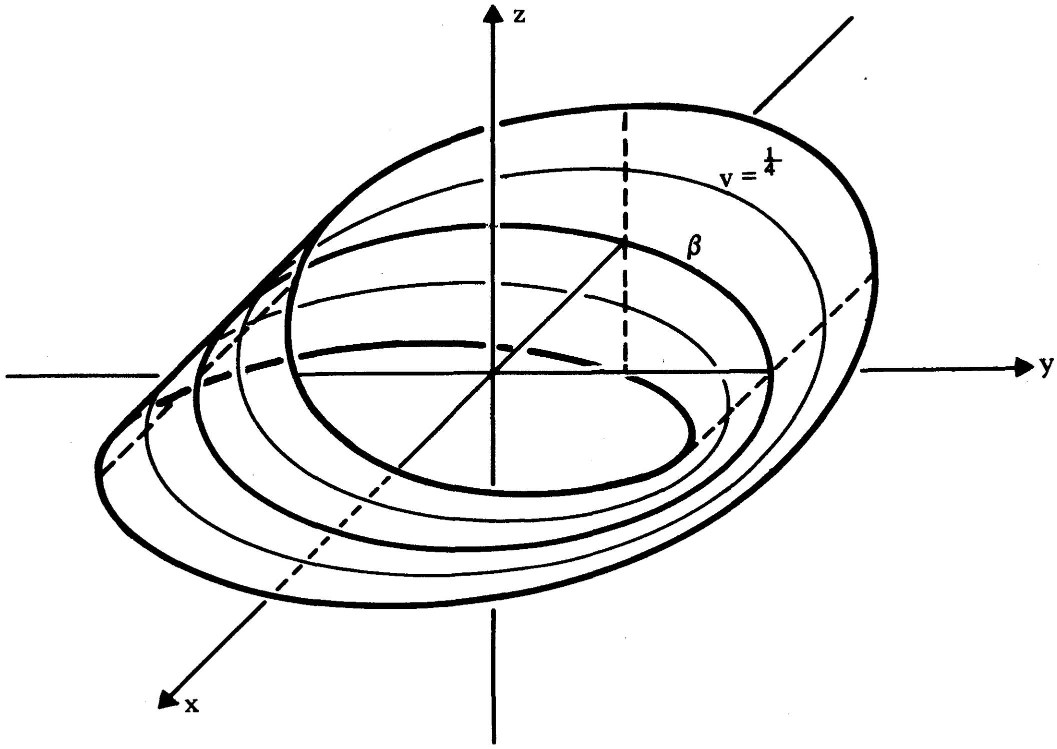
Other Recommended Exercises

Pages 180 - 181, Exercises 3, 9(b) and 9(c). These relate orientability to mappings.

Solutions

7(a). E, F, G were defined on page 140.

7(b).



The u-parameter curve $v = v_0 \neq 0$ is given by

$$u \longmapsto \beta(u) + v_0\,\delta(u).$$

Since β has period 2π and δ has period 4π this curve has period 4π.

8. M^* is the image, under x, of the region $0 < v$. In this region two points (u, v) and (u′, v′) give the same image under x if and only if $v = v'$ and $u = u' + 4\pi n$, for some integer n. The same is true of the vectors $x_u(u, v) \times x_v(u, v)$ and $x_u(u', v') \times x_v(u', v')$.

Hence $x_u \times x_v$ defines a normal vector field on M^*.

3. If μ is a never zero 2-form on N we prove that $F^*(\mu)$ is a never zero 2-form on M since

$$F^*(\mu)(x_u, x_v) = \mu(F_*(x_u), F_*(x_v)) \neq 0$$

since F is regular.

9(b). Let F map the Möbius band M (see Exercise 7) to the cylinder $N : x^2 + y^2 = 1$ as follows:

$$F : x(u, v) \longmapsto \beta(u) = (\cos u, \sin u, 0).$$

9(c). We want a regular mapping $F : M \longrightarrow N$ where M is orientable and N is not. Choose any patch on the Möbius band. For instance take $x : D \longrightarrow M$ where x is as in Exercise 7 and D is the set of points in E^2 such that $-\pi < u < \pi$, and $-\frac{1}{2} < v < \frac{1}{2}$. We know that E^2 can be considered as a surface in E^3.

DIFFERENTIAL GEOMETRY

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